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Differential Geometry

The tetrahedral property and a new Gromov–Hausdorff compactness theorem

La propriété tétraédrique et un nouvel théorème de compacité Gromov-Hausdorff

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ABSTRACT

We present the Tetrahedral Compactness Theorem, which states that sequences of Riemannian manifolds with a uniform upper bound on volume and diameter that satisfy a uniform tetrahedral property have a subsequence which converges in the Gromov-Hausdorff sense to a countably \mathcal{H}^m rectifiable metric space of the same dimension. The tetrahedral property depends only on distances between points in spheres; yet we show it provides a lower bound on the volumes of balls. The proof is based upon intrinsic flat convergence and a new notion called the sliced filling volume of a ball.

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RÉSUMÉ

Nous présentons le théorème tétraédrique de compacité, qui stipule que les séquences de variétés riemanniennes avec une borne supérieure uniforme sur le volume et sur le diamètre, qui satisfont une propriété tétraédrique uniforme, admettent une sous-suite qui converge, au sens de Gromov–Hausdorff, vers un espace métrique dénombrable \mathcal{H}^m , rectifiable, de la même dimension. La propriété tétraédrique ne dépend que de la distance entre les points dans les sphères, mais nous montrons qu'elle fournit une borne inférieure sur le volume des boules.

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Gromov's Compactness Theorem has transformed the field of Riemannian Geometry [3]. It states that a sequence of Riemannian manifolds, M_i^m , with uniform upper bounds on $\text{Diam}(M_i^m)$ and on the number, N(r), of disjoint balls of any given radius, r > 0, has a subsequence which converges in the Gromov–Hausdorff sense to the limit space, X: so there exist $\epsilon_i \rightarrow 0$ and maps $f_i : M_i^m \rightarrow X$ such that $|d_X(f_i(p), f_i(q)) - d_{M_i^m}(p, q)| < \epsilon_i$ and $X_i \subset T_{\epsilon_i}(f_i(M_i^m))$. Under these conditions the sequence of manifolds may collapse to a lower dimensional limit space which need not be rectifiable (cf. [5]).

Here we introduce a property we call the tetrahedral property which is a purely metric property guaranteeing that points lying on tetrahedra are kept a definite distance apart from one another (see Fig. 1) [Definition 1]. This tetrahedral property is strong enough to prevent the collapse of tetrahedra. Consequently we are not only able to prove a new Gromov–Hausdorff Compactness Theorem [Theorem 4] but also prove that the Gromov–Hausdorff limits obtained from sequences of manifolds satisfying this property do not collapse to a lower dimension and are in fact countably \mathcal{H}^m rectifiable [Remark 1].

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Fig. 1. Tetrahedral property in 3D.

Definition 1. Given C > 0 and $\beta \in (0, 1)$, a metric space *X* has the *m* dimensional *C*, β -tetrahedral property at a point *p* for radius *r* if one can find points $p_1, \ldots, p_{m-1} \subset \partial B_p(r) \subset \bar{X}$, such that

$$h(p, r, t_1, \dots, t_{m-1}) \ge Cr \quad \forall (t_1, \dots, t_{m-1}) \in \left[(1 - \beta)r, (1 + \beta)r \right]^m$$
(1)

where $h(p, r, t_1, ..., t_{m-1}) = \inf\{d(x, y): x \neq y, x, y \in P(p, r, t_1, ..., t_{m-1})\}$ when

$$P(p, r, t_1, \dots, t_{m-1}) = \rho_p^{-1}(r) \cap \rho_{p_1}^{-1}(t_1) \cap \dots \cap \rho_{p_{m-1}}^{-1}(t_{m-1}) \neq \emptyset$$
(2)

and $h(p, r, t_1, \dots, t_{m-1}) = 0$ otherwise. In particular $P(p, r, t_1, \dots, t_{m-1})$ is a discrete set of points.

In the next three examples we explore the meaning behind this definition:

Example 1. On Euclidean space, \mathbb{E}^3 , taking $p_1, p_2 \in \partial B(p, r)$ to such that $d(p_1, p_2) = r$, then there exist exactly two points $x, y \in P(p, r, r, r)$ each forming a tetrahedron with p, p_1, p_2 . As we vary $t_1, t_2 \in (r/2, 3r/2)$, we still have exactly two points in $P(p, r, t_1, t_2)$. By scaling we see that:

$$h(p, r, t_1, t_2) = rh(p, 1, t_1/r, t_2/r) \ge C_{\mathbb{R}^3} r$$
(3)

where $C_{\mathbb{E}^3} = \inf\{h(p, 1, s_1, s_2): s_i \in (1/2, 3/2)\} > 0$. Taking $\beta = 1/2$, we see that \mathbb{E}^3 satisfies the $C_{\mathbb{E}^3}$, β tetrahedral property.

Example 2. On a torus, $M_{\epsilon}^3 = S^1 \times S_{\epsilon}^1 \times S_{\epsilon}^1$ where S_{ϵ}^1 has been scaled to have diameter ϵ instead of π , we see that M^3 satisfies the $C_{\mathbb{E}^3}$, (1/2) tetrahedral property at p for all $r < \epsilon/4$. By taking $r < \epsilon/4$, we guarantee that the shortest paths between x and y stay within the ball B(p, r) allowing us to use the Euclidean estimates. If r is too large, $P(p, r, t_1, t_2) = \emptyset$. So for a sequence M_{ϵ} with $\epsilon \to 0$ we fail to have a uniform tetrahedral property. There is a Gromov–Hausdorff limit but it is not three dimensional.

Example 3. Suppose one creates a Riemannian manifold M_{ε}^3 , by gluing together two copies of Euclidean space with a large collection of tiny necks between corresponding points. That is,

$$M_{\varepsilon}^{3} = \left(\mathbb{E}^{3} \setminus \bigcup B_{Z_{i}}(\varepsilon)\right) \sqcup \left(\mathbb{E}^{3} \setminus \bigcup B_{Z_{i}}(\varepsilon)\right)$$

$$\tag{4}$$

where points on $\partial B_{z_i}(\varepsilon)$ in the first copy of Euclidean space are joined to corresponding points on $\partial B_{z_i}(\varepsilon_i)$ in the second copy of Euclidean space. We choose z_i such that $\mathbb{E}^3 \subset \bigcup_{i=1}^{\infty} B_{z_i}(10\varepsilon)$ and the balls $B_{z_i}(\varepsilon)$ are pairwise disjoint. Then for $r \gg \varepsilon$, we will have an x and a y as in \mathbb{E}^3 , but we will also have a nearby x' and y' in the second copy, with $d(x, x') < 20\varepsilon$. So for a sequence M_{ε} with $\varepsilon \to 0$ we fail to have a uniform tetrahedral property. If we create M_{ε}^3 by joining increasingly many copies of Euclidean space together, this sequence would not even have a subsequence converging in the Gromov-Hausdorff sense.

As this property is rather strong to use in some settings we introduce the integral tetrahedral property:

Definition 2. Given C > 0 and $\beta \in (0, 1)$, a metric space X is said to have the m dimensional integral C, β -tetrahedral property at a point p for radius r if $\exists p_1, \ldots, p_{m-1} \subset \partial B_p(r) \subset \overline{X}$, such that:

$$\int_{t_1=(1-\beta)r}^{(1+\beta)r} \cdots \int_{t_{m-1}=(1-\beta)r}^{(1+\beta)r} h(p,r,t_1,\ldots,t_{m-1}) dt_1 dt_2 \dots dt_{m-1} \ge C(2\beta)^{m-1} r^m.$$
(5)

Below we will describe how these properties provide the following estimate on volume:

Theorem 3. If p_0 lies in a Riemannian manifold that has the *m* dimensional (integral) *C*, β -tetrahedral property at a point *p* for radius *R* then Vol(B(p, r)) $\geq C(2\beta)^{m-1}r^m$.

As a consequence of Gromov's Compactness Theorem, we then have:

Theorem 4. Given $r_0 > 0$, $\beta \in (0, 1)$, C > 0, $V_0 > 0$. If a sequence of compact Riemannian manifolds, M^m , has $Vol(M^m) \leq V_0$, $Diam(M^m) \leq D_0$, and the C, β (integral) tetrahedral property for all balls of radius $\leq r_0$, then a subsequence converges in the Gromov–Hausdorff sense. In particular, they have a uniform upper bound on diameter depending only on these constants.

Remark 1. In fact we prove there is an intrinsic flat limit as well and the intrinsic flat and Gromov–Hausdorff limits agree. Thus the limit space in Theorem 4 is a countably \mathcal{H}^m rectifiable metric space.

The intrinsic flat distance between Riemannian manifolds was introduced by the author and Stefan Wenger in [6]. It was defined using Gromov's idea of isometrically embedding two Riemannian manifolds into a common metric space. Rather than measuring the Hausdorff distance between the images as Gromov did when defining the Gromov-Hausdorff distance in [3], one views the images as integral currents in the sense of Ambrosio–Kirchheim in [1] and takes the flat distance between them. The author and Wenger proved that intrinsic flat limit spaces are countably \mathcal{H}^m rectifiable metric spaces in [6].

Our compactness theorem is based upon the Gromov's Compactness Theorem [3] and the fact that we obtain a uniform lower bound on the volumes of balls [Theorem 3]. Applying Ambrosio–Kirchheim's Compactness Theorem of [1], Wenger and the author proved that once a sequence of manifolds converges in the Gromov–Hausdorff sense to a limit space Y, then a subsequence converges in the intrinsic flat sense to a subset, X, of Y [6]. In [5], estimates on the filling volumes of spheres were applied to prove the two limit spaces were the same when the sequence of manifolds has nonnegative Ricci curvature. Recall that filling volumes were introduced by Gromov in [2].

Here we do not have strong estimates on the filling volumes of spheres. To prove Theorems 3 and 4 we first define the sliced filling volumes of balls and then prove a new compactness theorem:

Definition 5. Given points $q_1, \ldots, q_k \in M^m$, where k < m, with distance functions $\rho_i(x) = d(x, q_i)$, we define the sliced filling volume of a sphere, $\partial B(p, r)$, to be:

$$\mathbf{SF}(p,r,q_1,\ldots,q_k) = \int_{t_1=m_1}^{M_1} \int_{t_2=m_2}^{M_2} \cdots \int_{t_k=m_k}^{M_k} \operatorname{FillVol}(\partial \operatorname{Slice}(B(p,r),\rho_1,\ldots,\rho_k,t_1,\ldots,t_k))\mathcal{L}^k$$
(6)

where $m_i = \min\{\rho_i(x): x \in \bar{B}_p(r)\}$ and $M_i = \max\{\rho_i(x): x \in \bar{B}_p(r)\}$ and where the slice is defined as in Geometric Measure Theory so that it is supported on $B(p, r) \cap \rho_1^{-1}(t_1) \cap \cdots \cap \rho_k^{-1}(t_k)$.

Definition 6. Given $p \in M^m$, then for almost every *r*, we can define the *k*th sliced filling,

$$\mathbf{SF}_{k}(p,r) = \sup \{ \mathbf{SF}(p,r,q_{1},\ldots,q_{k}) \colon q_{i} \in \partial B_{p}(r) \}.$$

$$\tag{7}$$

Theorem 7. Let V_0 , D_0 , $r_0 > 0$ and C(r) > 0. If M_i^m have $Vol(M_i) \leq V_0$, $Diam(M_i) \leq D_0$, and

$$\mathbf{SF}_{k}(p,r) \ge C(r) > 0 \quad \forall i \in \mathbb{N}, \ \forall p \in M_{i} \ and \ almost \ every \ r \in (0,r_{0})$$

$$\tag{8}$$

then a subsequence of the M_i converges in the Gromov–Hausdorff sense to a limit space which is also the intrinsic flat limit of the sequence and is thus a countably \mathcal{H}^m rectifiable metric space.

This theorem is proven by the author in [4]. We first show that $Vol(B(p, r)) \ge \mathbf{SF}_k(p, r)$, so that we can apply Gromov's Compactness Theorem to obtain a subsequence with a Gromov–Hausdorff limit, M_{∞} . We next prove that when the points $p_j \in M_j$ converge to a point p_{∞} in the Gromov–Hausdorff limit, M_{∞} , their sliced fillings converge. Applying Ambrosio–Kirchheim's Slicing Theorem, we can then estimate the mass of the limit current and prove that the Gromov–Hausdorff and intrinsic flat limits agree.

The final step in the proof of Theorem 4 is to relate the tetrahedral property to the k = m - 1 sliced filling volume. We first observe that:

$$\operatorname{spt}(\partial \operatorname{Slice}(B(p,r),\rho_1,\ldots,\rho_{m-1},t_1,\ldots,t_{m-1})) = \partial B_p(r) \cap \bigcap_{i=1}^{m-1} \partial B_{q_i}(t_i)$$
(9)

which is a discrete collection of points for almost every value of (t_1, \ldots, t_{m-1}) . So we prove a theorem that the filling volume of a 0 dimensional integral current can be bounded below by the distance between the closest pair of points in the current's support. We then have:

Theorem 8. If M^m is a Riemannian manifold with the *m* dimensional (integral) *C*, β -tetrahedral property at a point *p* for radius *r* then $Vol(B(p, r)) \ge SF_{m-1}(p, r) \ge C(2\beta)^{m-1}r^m$.

Combining this theorem with Theorem 7, we obtain both Theorem 4 and Remark 1.

These theorems and related theorems are proven by the author in full detail in [4], which is available on the arxiv. That paper will include many additional results before it is completed, as it explores many properties which are continuous under intrinsic flat convergence even in settings where there are no Gromov–Hausdorff limits and where the spaces are not Riemannian manifolds.

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