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Note on the completion of a local domain with geometrically normal formal fibers

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ARTICLE INFO	ABSTRACT
Article history: Received 7 December 2012 Accepted after revision 8 March 2013 Available online 10 April 2013 Presented by Michel Raynaud	In this note we will prove that if (R, m) is a local domain such that its formal fibers are geometrically normal, then the number of minimal prime ideals in the m-adic completion \widehat{R} equals exactly the number of maximal prime ideals in the integral closure \overline{R} of R in its field of quotients. © 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved. R É S U M É
	Dans cette note, nous allons montrer que, si (R, \mathfrak{m}) est un anneau local intègre tel que ses fibres formelles soient géométriquement normales, alors le nombre des idéaux premiers minimaux dans la complétion \mathfrak{m} -adique \widehat{R} égale exactement le nombre d'idéaux maximaux dans la clôture algébrique \overline{R} de R dans son corps de fractions. © 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

We assume that all rings mentioned in this note are commutative with identity. Let $\phi : R \longrightarrow A$ be a morphism of Noetherian rings. The fibers of ϕ are geometrically normal (resp. geometrically regular) if for every field K, which is a localization of a finitely generated R-algebra, the ring $K \otimes_R A$ is normal (resp. regular). The map ϕ is called regular if it is flat and its fibers are geometrically regular. Let (R, m) be a Noetherian local ring and \widehat{R} the m-adic completion of R. The formal fibers of R are geometrically normal (resp. geometrically regular) if the fibers of the completion map $R \longrightarrow \widehat{R}$ are geometrically normal (resp. geometrically regular). A Noetherian local ring is quasi-excellent (called also G-ring) if its formal fibers are geometrically regular.

Let now (R, m) be a Noetherian local domain ring and R^h (resp. \overline{R}) the Henselization of R (resp. the integral closure R in its field of quotients). By Nagata's Theorem [6, (43.20)], there exists an one-to-one correspondence between the maximal ideals in \overline{R} and the minimal prime ideals in R^h . When (R, m) is quasi-excellent, then R^h is also quasi-excellent, so in this case the number of minimal prime ideals of \widehat{R} equals the number of maximal prime ideals in \overline{R} , we can obtain this from the Artin Approximation Theorem. Indeed, we can assume that R^h is an integral domain, and we will show that \widehat{R} is again an integral domain. If there is an equation in \widehat{R} of the form XY = 0 that has a non-trivial solution, we can approximate this solution in the m-adic topology by a solution in R^h , and this is a contradiction with the assumption that R^h is an integral domain. There is another proof of this equality without using the Artin Approximation Theorem and for more details about this proof we refer the reader to [1, Theorem 6.5, p. 119].

In general, if (R, \mathfrak{m}) is a Noetherian local ring, the number of minimal prime ideals in \widehat{R} is greater than or equals the number of minimal prime ideals in R^h . In fact, any minimal prime ideal of \widehat{R} is contracted to a minimal prime ideal

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of R^h , but it is possible that two distinct minimal prime ideals of \widehat{R} can contract to the same minimal prime ideal of R^h . An exercise in [6, Exercise 1, p. 122], answered by Katz [3, Corollary 5], shows that if R is an one-dimensional local domain, then the number of minimal prime ideals in \widehat{R} is equal to the number of maximal ideals in \overline{R} .

Our objective is to prove the equality between the number of minimal prime ideals in \widehat{R} and the number of maximal prime ideals in the integral closure \overline{R} of R in its total ring of fractions in the case where the formal fibers of R are geometrically normal (see Theorem 2.3). This result shows that the hypothesis of quasi-excellence of R is not necessary to obtain this equality; it suffices just to suppose that its formal fibers are geometrically normal.

2. Main result

To present the main result of this note, we need to start with the following lemma inspired form [2].

Lemma 2.1. Let (R, \mathfrak{m}) be a local domain such that its formal fibers are geometrically normal. Then the following conditions are equivalent:

- (1) \overline{R} is local,
- (2) *R* is analytically irreducible (i.e. \widehat{R} is an integral domain),
- (3) R^h is an integral domain.

Proof. The natural inclusion $i : R \longrightarrow R^h$ induces an isomorphism between \widehat{R} and $\widehat{R^h}$. Consequently, if R is analytically irreducible, then R^h is also analytically irreducible, and hence R^h is an integral domain as a subring of its completion. So we have $(2) \Longrightarrow (3)$. The implication $(1) \Longrightarrow (2)$ can be obtained from [2, Proposition 2.2]. In fact, assume that \overline{R} is local, and denote by $\overline{\mathfrak{m}}$ its unique maximal ideal. Since the fibers of R are geometrically normal, then it is the same for the formal fibers of \overline{R} . Hence the $\overline{\mathfrak{m}}$ -adic completion $\widehat{\overline{R}}$ of \overline{R} is normal [5, p. 185]. On the other hand, R is analytically unramified. Indeed, $k(\mathfrak{p}) \otimes \widehat{R}$ is normal for all prime ideal \mathfrak{p} of R, hence reduced, and by [5, Theorem 23.9], we obtain that \widehat{R} satisfies Serre's conditions (S_1) and (R_0) , therefore \widehat{R} is reduced. Analogously, we prove that $\widehat{\overline{R}}$ is reduced. Therefore, \overline{R} is analytically irreducible [4, (3.5) Corollary, p. 422]. Then \overline{R} is finitely generated over R [6, (32.2)] and by using [6, (17.8), p. 56], we obtain $\widehat{\overline{R}} = \widehat{R} \otimes_R \overline{R}$, then \widehat{R} is an integral domain. The implication (3) \Longrightarrow (1) is a special case of Nagata's Theorem [6, (43.20)].

Lemma 2.2. Let *R* be a Noetherian local ring such that every formal fiber of *R* is a disjoint union of a finite number of integral schemes (i.e. for all prime ideal p in *R* and for all prime ideal q in \hat{R} such that $q \cap R = p$, the ring $R_q/p\hat{R}_q$ is a domain). Then the same holds for the formal fibers of R^h . In particular, if the formal fibers of *R* are geometrically normal, then the formal fibers of R^h are also geometrically normal.

Proof. Let \mathfrak{p}^h be a prime ideal in \mathbb{R}^h and \mathfrak{q} a prime ideal in $\widehat{\mathbb{R}}$ such that $\mathfrak{q} \cap \mathbb{R}^h = \mathfrak{p}^h$. Set $\mathfrak{p} = \mathfrak{p}^h \cap \mathbb{R}$, then we have:

 $\mathfrak{p} = \mathfrak{q} \cap R$ and $\mathfrak{p}\widehat{R}_{\mathfrak{q}} \subseteq \mathfrak{p}^h\widehat{R}_{\mathfrak{q}}$.

By hypothesis, the ideal $\mathfrak{p}\widehat{R}_{\mathfrak{q}}$ is prime. Since $\mathfrak{p}\widehat{R}_{\mathfrak{q}}$ and $\mathfrak{p}^{h}\widehat{R}_{\mathfrak{q}}$ have the same height, we obtain $\mathfrak{p}\widehat{R}_{\mathfrak{q}} = \mathfrak{p}^{h}\widehat{R}_{\mathfrak{q}}$. Hence $\widehat{R}_{\mathfrak{q}}/\mathfrak{p}^{h}\widehat{R}_{\mathfrak{q}}$ is a domain. \Box

Theorem 2.3. Let (R, \mathfrak{m}) be a local domain such that its formal fibers are geometrically normal. Then the number of minimal prime ideals in the \mathfrak{m} -adic completion \widehat{R} of R equals exactly the number of maximal prime ideals in the integral closure \overline{R} of R in its field of fractions.

Proof. We know that the number of minimal prime ideals of \mathbb{R}^h coincides with the number of maximal ideals in $\overline{\mathbb{R}}$. To prove this theorem, it remains to show that the number of minimal prime ideals of \mathbb{R}^h equals the number of minimal prime ideals of $\widehat{\mathbb{R}}$. Since the formal fibers of \mathbb{R} are geometrically normal, then the formal fibers of its Henselization \mathbb{R}^h are also geometrically normal (see Lemma 2.2). Let \mathfrak{p} be a minimal prime ideal in \mathbb{R}^h . Then $\mathbb{R}^h/\mathfrak{p}$ is a local Henselian domain with geometrically normal formal fibers. From Lemma 2.1, the $\mathfrak{m}\mathbb{R}^h/\mathfrak{p}$ -adic completion of $\mathbb{R}^h/\mathfrak{p}$ is also an integral domain, that means:

$$\widehat{(R^h/\mathfrak{p})} = \widehat{R}/\mathfrak{p}\widehat{R}$$

is an integral domain. So $\mathfrak{p}\widehat{R}$ is necessarily a minimal prime ideal in \widehat{R} . Hence we have an one to one correspondence between minimal prime ideals of R^h and the minimal prime ideals of \widehat{R} . \Box

3. Conclusion

The main result of this note shows that the assumption: *R* is quasi-excellent (i.e. its formal fibers are geometrically regular) is not necessary to obtain the equality between the number of minimal prime ideals in \widehat{R} and the number of maximal prime ideals in \overline{R} , it suffices to suppose just that the formal fibers of *R* are geometrically normal.

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