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Number Theory

On a theorem of Kisin

Sur un théorème de Kisin

Alexander Beilinson^{a,1}, Floric Tavares Ribeiro^b

^a Department of Mathematics, University of Chicago, Chicago, IL 60637, USA

^b Normandie Université, LMNO, CNRS UMR 6139, 14032 Caen, France

ARTICLE INFO	A B S T R A C T
Article history: Received 13 March 2013 Accepted after revision 2 April 2013 Available online 2 August 2013 Presented by Jean-Marc Fontaine	This note provides a short proof of a theorem of Kisin on crystalline representations. © 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved. R É S U M É
	Dans cette note, on donne une preuve courte d'un théorème de Kisin sur les représenta- tions cristallines. © 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Let *K* be a *p*-adic field, i.e., a complete discretely-valued field of characteristic 0 with perfect residue field of characteristic p > 0, and \bar{K} be an algebraic closure of *K*. We fix a uniformizer $\pi \in K$. Let $\Xi = \Xi_{\pi}$ be the corresponding Kummer $\mathbb{Z}_p(1)$ -torsor; its elements are sequences $\xi = (\xi_n)_{n \ge 0}$ of elements in \bar{K} such that $\xi_{n+1}^p = \xi_n$, $\xi_0 = \pi$. Pick one ξ , and set $K_{\xi} = \bigcup K(\xi_n)$. Consider the Galois groups $G := \operatorname{Gal}(\bar{K}/K)$, $G_{\xi} := \operatorname{Gal}(\bar{K}/K_{\xi})$; let $\operatorname{Rep}(G)$, $\operatorname{Rep}(G_{\xi})$ be the categories of their finite-dimensional \mathbb{Q}_p -representations.

The next result was conjectured by Breuil [1] and proved by Kisin [4, 0.2]; the proof in [4] is based on the theory of Kisin modules. This note provides an alternative argument that uses only basic properties of Fontaine's rings; its key ingredient (namely, (i) of the lemma below) is the same as in Grothendieck's proof of the monodromy theorem.

Theorem. The restriction functor $\operatorname{Rep}(G) \to \operatorname{Rep}(G_{\xi})$ is fully faithful on the subcategory of crystalline representations.

Proof. The Galois group *G* acts on Ξ , and G_{ξ} is the stabilizer of ξ . The action is transitive, i.e., $G/G_{\xi} \xrightarrow{\sim} \Xi$, since polynomials $t^{p^n} - \pi$ are irreducible. Let *R* be the ring of continuous \mathbb{Q}_p -valued functions on Ξ . Let $R_{st} \subset R_{\phi}$ be the subrings of polynomial, resp. locally polynomial, functions (this makes sense since Ξ is $\mathbb{Z}_p(1)$ -torsor). Since *G* acts on Ξ by affine transformations, its action on *R* preserves the subrings.

Lemma.

- (i) R_{ϕ} is the union of all finite-dimensional *G*-submodules of *R*.
- (ii) R_{st} is the union of all semi-stable *G*-submodules of R_{ϕ} .
- (iii) \mathbb{Q}_p is the only nontrivial crystalline *G*-submodule of R_{st} .

E-mail addresses: sasha@math.uchicago.edu (A. Beilinson), floric.tavares-ribeiro@unicaen.fr (F. Tavares Ribeiro).

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Assuming the lemma, let us prove the theorem. For $V \in \operatorname{Rep}(G_{\xi})$ we denote by I(V) the induced *G*-module. Thus I(V) is the space of all continuous maps $f: G \to V$ such that f(hg) = hf(g) for $h \in G_{\xi}$, the action of *G* is g(f)(g') = f(g'g). For $U \in \operatorname{Rep}(G)$ we have the Frobenius reciprocity $\operatorname{Hom}_{G_{\xi}}(U, V) \xrightarrow{\sim} \operatorname{Hom}_{G}(U, I(V))$ that identifies $\alpha : U \to V$ with $\tilde{\alpha} : U \to I(V)$, $\tilde{\alpha}(u)(g) = \alpha(gu), \alpha(u) = \tilde{\alpha}(u)(1)$. For $V \in \operatorname{Rep}(G)$ the image of $\operatorname{id}_{V} \in \operatorname{Hom}_{G_{\xi}}(V, V)$ is a *G*-morphism $V \to I(V)$ that yields an identification of *G*-equivariant *R*-modules $V \otimes R \xrightarrow{\sim} I(V)$.

So for $V_1, V_2 \in \text{Rep}(G)$ one has identifications $\text{Hom}_{G_{\xi}}(V_1, V_2) = \text{Hom}_G(V_1, I(V_2)) = \text{Hom}_G(V_1, V_2 \otimes R) = \text{Hom}_G(V_1 \otimes V_2^*, R) = \text{Hom}_G(V_1 \otimes V_2^*, R_{\phi})$, the last equality comes from (i). If both V_i are crystalline, then this equals $\text{Hom}_G(V_1 \otimes V_2^*, \mathbb{Q}_p) = \text{Hom}_G(V_1, V_2)$ by (ii), (iii). Thus $\text{Hom}_{G_{\xi}}(V_1, V_2) = \text{Hom}_G(V_1, V_2)$. \Box

Proof of Lemma. Let *P* be the group of all affine automorphisms of $\mathbb{Z}_p(1)$ -torsor Ξ ; it is an extension of \mathbb{Z}_p^{\times} by $\mathbb{Z}_p(1)$, the choice of ξ gives a splitting. Let $\eta : G \to P$ be the action of *G* on Ξ ; its composition with $P \twoheadrightarrow \mathbb{Z}_p^{\times}$ is the cyclotomic character χ .

Consider the filtration R_{stn} on R_{st} by the degree of the polynomial. Then G acts on $gr_n R_{st}$ by χ^{-n} , i.e., $gr_n R_{st}$ is isomorphic to $\mathbb{Q}_p(-n)$.

There is a canonical morphism $\varepsilon : R_{st} \to B_{st}$ of \mathbb{Q}_p -algebras defined as follows. For $\xi \in \Xi$ let $l_{\xi} : \Xi \to \mathbb{Z}_p(1)$ be the identification of torsors such that $l_{\xi}(\xi) = 0$. If τ is a generator of $\mathbb{Z}_p(1)$, then $\tau^{-1}l_{\xi} \in R_{st}$ is a linear polynomial function, i.e., a free generator of R_{st} . We define ε by formula $\varepsilon(\tau^{-1}l_{\xi}) = -\tau^{-1}\lambda(\xi)$. Here in the r.h.s. we view τ as an invertible element of B_{crys} via the embedding $\mathbb{Z}_p(1) \hookrightarrow B_{crys}$ from [2, 2.3.4], and $\lambda(\xi) \in B_{st}$ is as in [2, 3.1.4]. It follows from the definitions in [2, 3.1] that ε does not depend on the auxiliary choice of ξ . It evidently commutes with the Galois action. Since $\log(\xi)$ is a free generator of B_{st} over B_{crys} , we see that ε is injective and R_{stn} for $n \ge 1$ are *non-crystalline* semi-stable *G*-modules.

Choose v and log from [3, 5.1.2] as $v(\pi) = 1$, $\log(\pi) = 0$. As in [3, 5.2], this yields the fully faithful tensor functor D_{st} : $\operatorname{Rep}(G)_{st} \to \operatorname{MF}_{K}(\varphi, N)$.

Consider the polynomial algebra $K_0[t]$. We equip it with Frobenius semi-linear automorphism φ , $\varphi(t) := pt$, the K_0 -derivation $N := \partial_t$, and the Hodge filtration $F^i :=$ the K-span of $t^{\geq i}$. The subspaces of polynomials of degree $\leq n$ are filtered (φ, N) -modules, so $K_0[t]$ is a ring ind-object of $MF_K(\varphi, N)$.

There is a canonical isomorphism $K_0[t] \xrightarrow{\sim} D_{st}(R_{st})$ which identifies t with $(\tau^{-1}l_{\xi}) \otimes \tau + 1 \otimes \lambda(\xi) \in (R_{st} \otimes B_{st})^G = D_{st}(R_{st})$. Thus each $D_{st}(R_{stn})$ is a single Jordan block for the action of N, so every finite-dimensional G-submodule of R_{st} equals one of R_{stn} 's, which implies (iii).

Notice that $R_{\phi} = R_0 \otimes R_{st}$, where R_0 is the subring of locally constant functions. Since *G* acts transitively on Ξ , one has $R_0^G = \mathbb{Q}_p$ and all finite-dimensional *G*-modules that occur in R_0 are generated by G_{ξ} -fixed vectors. These representations are Artinian, hence semisimple, so we have the decomposition $R_0 = \mathbb{Q}_p \oplus R'_0$. Since the map $G_{\xi} \to \text{Gal}(K^{\text{un}}/K)$, where $K^{\text{un}} \subset \overline{K}$ is the maximal unramified extension of *K*, is surjective (for $K^{\text{un}} \cap K_{\xi} = K$), every *G*-module in R'_0 is ramified. Thus every irreducible subquotient of $R'_0 \otimes R_{st}$ is *not* semi-stable, and we get (ii).

It remains to prove (i). We first show that $\eta(G)$ is an open subgroup of *P*. Since $\chi(G)$ is an open subgroup of \mathbb{Z}_p^{\times} , it suffices to check that $\eta(G) \cap \mathbb{Z}_p(1)$ is open in $\mathbb{Z}_p(1)$. Since every closed nontrivial subgroup of $\mathbb{Z}_p(1)$ is open, we need to check that $\eta(G) \cap \mathbb{Z}_p(1) \neq \{0\}$. If not, then $\eta(G) \xrightarrow{\sim} \chi(G)$ is commutative, so *G* acts on *R* through an abelian quotient. This implies, since $\operatorname{gr}_n R_{\operatorname{st}} \simeq \mathbb{Q}_p(-n)$ are pairwise non-isomorphic *G*-modules, that filtration R_{stn} splits, which is not true, q.e.d.

Let τ be a generator of $\mathbb{Z}_p(1) \subset P$; then R_{ϕ} is the union of all finite-dimensional $\mathbb{Z}_p(1)$ -submodules of R on which all eigenvalues of τ are roots of 1. Since $\eta(G)$ has finite index in P, it suffices to show that every finite-dimensional Psubmodule V of R has this property. This follows since for $g \in P$ one has $g\tau g^{-1} = \tau^m$, where m is the image of g in \mathbb{Z}_p^{\times} , and there are only finitely many eigenvalues of τ on V. \Box

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