Number Theory

## On a theorem of Kisin

## Sur un théorème de Kisin

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## A B S T R A C T

This note provides a short proof of a theorem of Kisin on crystalline representations.
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R É S U M É

Dans cette note, on donne une preuve courte d'un théorème de Kisin sur les représentations cristallines.
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Let $K$ be a $p$-adic field, i.e., a complete discretely-valued field of characteristic 0 with perfect residue field of characteristic $p>0$, and $\bar{K}$ be an algebraic closure of $K$. We fix a uniformizer $\pi \in K$. Let $\Xi=\Xi_{\pi}$ be the corresponding Kummer $\mathbb{Z}_{p}(1)-$ torsor; its elements are sequences $\xi=\left(\xi_{n}\right)_{n \geqslant 0}$ of elements in $\bar{K}$ such that $\xi_{n+1}^{p}=\xi_{n}, \xi_{0}=\pi$. Pick one $\xi$, and set $K_{\xi}=$ $\bigcup K\left(\xi_{n}\right)$. Consider the Galois groups $G:=\operatorname{Gal}(\bar{K} / K), G_{\xi}:=\operatorname{Gal}\left(\bar{K} / K_{\xi}\right)$; let $\operatorname{Rep}(G), \operatorname{Rep}\left(G_{\xi}\right)$ be the categories of their finitedimensional $\mathbb{Q}_{p}$-representations.

The next result was conjectured by Breuil [1] and proved by Kisin [4, 0.2]; the proof in [4] is based on the theory of Kisin modules. This note provides an alternative argument that uses only basic properties of Fontaine's rings; its key ingredient (namely, (i) of the lemma below) is the same as in Grothendieck's proof of the monodromy theorem.

Theorem. The restriction functor $\operatorname{Rep}(G) \rightarrow \operatorname{Rep}\left(G_{\xi}\right)$ is fully faithful on the subcategory of crystalline representations.

Proof. The Galois group $G$ acts on $\Xi$, and $G_{\xi}$ is the stabilizer of $\xi$. The action is transitive, i.e., $G / G_{\xi} \xrightarrow{\sim} \Xi$, since polynomials $t^{p^{n}}-\pi$ are irreducible. Let $R$ be the ring of continuous $\mathbb{Q}_{p}$-valued functions on $\Xi$. Let $R_{\text {st }} \subset R_{\phi}$ be the subrings of polynomial, resp. locally polynomial, functions (this makes sense since $\Xi$ is $\mathbb{Z}_{p}(1)$-torsor). Since $G$ acts on $\Xi$ by affine transformations, its action on $R$ preserves the subrings.

## Lemma.

(i) $R_{\phi}$ is the union of all finite-dimensional $G$-submodules of $R$.
(ii) $R_{\text {st }}$ is the union of all semi-stable $G$-submodules of $R_{\phi}$.
(iii) $\mathbb{Q}_{p}$ is the only nontrivial crystalline $G$-submodule of $R_{\mathrm{st}}$.

[^0]Assuming the lemma, let us prove the theorem. For $V \in \operatorname{Rep}\left(G_{\xi}\right)$ we denote by $I(V)$ the induced $G$-module. Thus $I(V)$ is the space of all continuous maps $f: G \rightarrow V$ such that $f(h g)=h f(g)$ for $h \in G_{\xi}$, the action of $G$ is $g(f)\left(g^{\prime}\right)=f\left(g^{\prime} g\right)$. For $U \in \operatorname{Rep}(G)$ we have the Frobenius reciprocity $\operatorname{Hom}_{G \xi}(U, V) \xrightarrow{\sim} \operatorname{Hom}_{G}(U, I(V))$ that identifies $\alpha: U \rightarrow V$ with $\tilde{\alpha}: U \rightarrow I(V)$, $\tilde{\alpha}(u)(g)=\alpha(g u), \alpha(u)=\tilde{\alpha}(u)(1)$. For $V \in \operatorname{Rep}(G)$ the image of $\operatorname{id}_{V} \in \operatorname{Hom}_{G_{\xi}}(V, V)$ is a $G$-morphism $V \rightarrow I(V)$ that yields an identification of $G$-equivariant $R$-modules $V \otimes R \xrightarrow{\sim} I(V)$.

So for $V_{1}, V_{2} \in \operatorname{Rep}(G)$ one has identifications $\operatorname{Hom}_{G \xi}\left(V_{1}, V_{2}\right)=\operatorname{Hom}_{G}\left(V_{1}, I\left(V_{2}\right)\right)=\operatorname{Hom}_{G}\left(V_{1}, V_{2} \otimes R\right)=\operatorname{Hom}_{G}\left(V_{1} \otimes\right.$ $\left.V_{2}^{*}, R\right)=\operatorname{Hom}_{G}\left(V_{1} \otimes V_{2}^{*}, R_{\phi}\right)$, the last equality comes from (i). If both $V_{i}$ are crystalline, then this equals $\operatorname{Hom}_{G}\left(V_{1} \otimes\right.$ $\left.V_{2}^{*}, \mathbb{Q}_{p}\right)=\operatorname{Hom}_{G}\left(V_{1}, V_{2}\right)$ by (ii), (iii). Thus $\operatorname{Hom}_{G_{\xi}}\left(V_{1}, V_{2}\right)=\operatorname{Hom}_{G}\left(V_{1}, V_{2}\right)$.

Proof of Lemma. Let $P$ be the group of all affine automorphisms of $\mathbb{Z}_{p}(1)$-torsor $\Xi$; it is an extension of $\mathbb{Z}_{p}^{\times}$by $\mathbb{Z}_{p}(1)$, the choice of $\xi$ gives a splitting. Let $\eta: G \rightarrow P$ be the action of $G$ on $\Xi$; its composition with $P \rightarrow \mathbb{Z}_{p}^{\times}$is the cyclotomic character $\chi$.

Consider the filtration $R_{\text {st } n}$ on $R_{\text {st }}$ by the degree of the polynomial. Then $G$ acts on $\operatorname{gr}_{n} R_{\text {st }}$ by $\chi^{-n}$, i.e., $\operatorname{gr}_{n} R_{\text {st }}$ is isomorphic to $\mathbb{Q}_{p}(-n)$.

There is a canonical morphism $\varepsilon: R_{\mathrm{st}} \rightarrow \mathrm{B}_{\mathrm{st}}$ of $\mathbb{Q}_{p}$-algebras defined as follows. For $\xi \in \Xi$ let $l_{\xi}: \Xi \rightarrow \mathbb{Z}_{p}$ (1) be the identification of torsors such that $l_{\xi}(\xi)=0$. If $\tau$ is a generator of $\mathbb{Z}_{p}(1)$, then $\tau^{-1} l_{\xi} \in R_{\text {st }}$ is a linear polynomial function, i.e., a free generator of $R_{\mathrm{st}}$. We define $\varepsilon$ by formula $\varepsilon\left(\tau^{-1} l_{\xi}\right)=-\tau^{-1} \lambda(\xi)$. Here in the r.h.s. we view $\tau$ as an invertible element of $\mathrm{B}_{\text {crys }}$ via the embedding $\mathbb{Z}_{p}(1) \hookrightarrow \mathrm{B}_{\text {crys }}$ from [2, 2.3.4], and $\lambda(\xi) \in \mathrm{B}_{\text {st }}$ is as in [2, 3.1.4]. It follows from the definitions in [2,3.1] that $\varepsilon$ does not depend on the auxiliary choice of $\xi$. It evidently commutes with the Galois action. Since $\log (\xi)$ is a free generator of $\mathrm{B}_{\text {st }}$ over $\mathrm{B}_{\text {crys }}$, we see that $\varepsilon$ is injective and $R_{\text {st } n}$ for $n \geqslant 1$ are non-crystalline semi-stable $G$-modules.

Choose $v$ and $\log$ from [3,5.1.2] as $v(\pi)=1, \log (\pi)=0$. As in [3,5.2], this yields the fully faithful tensor functor $D_{\text {st }}: \operatorname{Rep}(G)_{\text {st }} \rightarrow \operatorname{MF}_{K}(\varphi, N)$.

Consider the polynomial algebra $K_{0}[t]$. We equip it with Frobenius semi-linear automorphism $\varphi, \varphi(t):=p t$, the $K_{0}-$ derivation $N:=\partial_{t}$, and the Hodge filtration $F^{i}:=$ the $K$-span of $t \geqslant i$. The subspaces of polynomials of degree $\leqslant n$ are filtered $(\varphi, N)$-modules, so $K_{0}[t]$ is a ring ind-object of $\operatorname{MF}_{K}(\varphi, N)$.

There is a canonical isomorphism $K_{0}[t] \xrightarrow{\sim} D_{\mathrm{st}}\left(R_{\mathrm{st}}\right)$ which identifies $t$ with $\left(\tau^{-1} l_{\xi}\right) \otimes \tau+1 \otimes \lambda(\xi) \in\left(R_{\mathrm{st}} \otimes \mathrm{B}_{\mathrm{st}}\right)^{G}=D_{\mathrm{st}}\left(R_{\mathrm{st}}\right)$. Thus each $D_{\mathrm{st}}\left(R_{\mathrm{st} n}\right)$ is a single Jordan block for the action of $N$, so every finite-dimensional $G$-submodule of $R_{\text {st }}$ equals one of $R_{\text {stn }}$ 's, which implies (iii).

Notice that $R_{\phi}=R_{0} \otimes R_{\text {st }}$, where $R_{0}$ is the subring of locally constant functions. Since $G$ acts transitively on $\Xi$, one has $R_{0}^{G}=\mathbb{Q}_{p}$ and all finite-dimensional $G$-modules that occur in $R_{0}$ are generated by $G_{\xi}$-fixed vectors. These representations are Artinian, hence semisimple, so we have the decomposition $R_{0}=\mathbb{Q}_{p} \oplus R_{0}^{\prime}$. Since the map $G_{\xi} \rightarrow \operatorname{Gal}\left(K^{\text {un }} / K\right)$, where $K^{\text {un }} \subset \bar{K}$ is the maximal unramified extension of $K$, is surjective (for $K^{\mathrm{un}} \cap K_{\xi}=K$ ), every $G$-module in $R_{0}^{\prime}$ is ramified. Thus every irreducible subquotient of $R_{0}^{\prime} \otimes R_{\mathrm{st}}$ is not semi-stable, and we get (ii).

It remains to prove (i). We first show that $\eta(G)$ is an open subgroup of $P$. Since $\chi(G)$ is an open subgroup of $\mathbb{Z}_{p}^{\times}$, it suffices to check that $\eta(G) \cap \mathbb{Z}_{p}(1)$ is open in $\mathbb{Z}_{p}(1)$. Since every closed nontrivial subgroup of $\mathbb{Z}_{p}(1)$ is open, we need to check that $\eta(G) \cap \mathbb{Z}_{p}(1) \neq\{0\}$. If not, then $\eta(G) \xrightarrow{\sim} \chi(G)$ is commutative, so $G$ acts on $R$ through an abelian quotient. This implies, since $\mathrm{gr}_{n} R_{\mathrm{st}} \simeq \mathbb{Q}_{p}(-n)$ are pairwise non-isomorphic $G$-modules, that filtration $R_{\text {st } n}$ splits, which is not true, q.e.d.

Let $\tau$ be a generator of $\mathbb{Z}_{p}(1) \subset P$; then $R_{\phi}$ is the union of all finite-dimensional $\mathbb{Z}_{p}(1)$-submodules of $R$ on which all eigenvalues of $\tau$ are roots of 1 . Since $\eta(G)$ has finite index in $P$, it suffices to show that every finite-dimensional $P$ submodule $V$ of $R$ has this property. This follows since for $g \in P$ one has $g \tau g^{-1}=\tau^{m}$, where $m$ is the image of $g$ in $\mathbb{Z}_{p}^{\times}$, and there are only finitely many eigenvalues of $\tau$ on $V$.

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