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Functional Analysis

# On T. Bartoszynski's structure theorem for measurable filters

## *Une preuve de la caractérisation par T. Bartoszynski des filtres mesurables*

Michel Talagrand

*Institut de Mathématiques, UMR 7586 CNRS, 4, place Jussieu, 75230 Paris cedex 05, France*

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ABSTRACT

We give a streamlined proof of T. Bartoszynski's characterization of Lebesgue-measurable filters.



RÉSUMÉ

Nous donnons une démonstration simplifiée d'un théorème remarquable de T. Bartoszynski caractérisant les filtres qui sont Lebesgue-mesurables en tant que sous-ensembles de $\{0,1\}^{\mathbb{N}}$.



### 1. Introduction

In this paper, we identify the collection of subsets of $\mathbb{N}$ with $\Omega := \{0,1\}^{\mathbb{N}}$; so a subset of $\mathbb{N}$ is denoted by $x$, $y$, $z$. A (proper) filter $\mathcal{F}$ is a collection of infinite subsets of $\mathbb{N}$ such that:

$$x \in \mathcal{F}, \quad x \subset y \quad \Rightarrow \quad y \in \mathcal{F},$$

$$x, y \in \mathcal{F} \quad \Rightarrow \quad x \cap y \in \mathcal{F}$$

and $[n,\infty[\, \in \mathcal{F}$ for each $n$. We denote by $\lambda$ the canonical measure on $\Omega$. By the zero–one law any filter is either of measure zero (and then measurable) or of outer measure 1 (and hence non-measurable).

Given a finite subset $I$ of $\mathbb{N}$ and $x \in \Omega$, we write

$$U(x, I) = \{y \in \Omega;\ \forall i \in I,\ y_i = x_i\}.$$

Then $\lambda(U(x,I)) = 2^{-\operatorname{card} I}$. There are exactly $2^{\operatorname{card} I}$ sets of this type, which form a partition of $\Omega$. We say that a subset $C$ of $\Omega$ depends only on the coordinates in $I$ if $C = \bigcup_{x \in C} U(x,I)$. Then given $x, y \in \Omega$ with $x_i = y_i$ for $i \in I$, either both of them or none of them belong(s) to $C$. The main purpose of this note is to give a streamlined proof of the following remarkable result of T. Bartoszynski [1].

**Theorem 1.** *A filter is measurable if and only if one can find disjoint finite sets $I_k$ and sets $C_k$ depending only on the coordinates in $I_k$ such that*:

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*E-mail address:* talagran@math.jussieu.fr.

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$$\sum_k \lambda(C_k) < \infty$$

*and each element $x \in \mathcal{F}$ belongs to infinitely many sets $C_k$.*

## 2. Proof of Theorem 1

The "if" part is trivial, and the problem is to prove the other direction. Given a compact set $K \subset \Omega$, we denote

$$K_n = \{y \in \Omega;\ \exists x \in K,\ \forall i < n,\ x_i = y_i\}; \qquad K^n = \{y \in \Omega;\ \exists x \in K,\ \forall i \geqslant n,\ x_i = y_i\}.$$

Thus $K = \bigcap_n K_n$; therefore

$$\lim_{n\to\infty} \lambda(K_n \setminus K) = 0.$$

Moreover, it is well known that:

$$\lambda(K) > 0 \quad \Rightarrow \quad \lim_{n\to\infty} \lambda\big(K^n\big) = 1.$$

Proving this property is how one may prove the zero–one law. We then denote $K_n^\ell = (K^\ell)_n$ and we observe that $K_n^n = \Omega$.

Let us set $n_0 = 0$. We then construct inductively a sequence $(n_k)$ growing fast enough so that:

$$p \leqslant k \quad \Rightarrow \quad \lambda\big(K_{n_{k+1}}^{n_p} \setminus K^{n_p}\big) \leqslant 2^{-n_k-3k-3}, \qquad \lambda\big(\Omega \setminus K^{n_{k+1}}\big) \leqslant 2^{-n_k-3k-3}.$$

We set

$$A_k = \bigcup_{p\leqslant k} \big(K_{n_k}^{n_p} \setminus K_{n_{k+1}}^{n_p}\big).$$

Since $A_k \subset (\Omega \setminus K^{n_k}) \cup \bigcup_{p<k} (K_{n_k}^{n_p} \setminus K^{n_p})$, we have $\lambda(A_k) \leqslant 2^{-n_{k-1}-2k}$. Moreover, the set $A_k$ depends only on the coordinates of rank $< n_{k+1}$. We consider the disjoint intervals $I_k = [n_k, n_{k+1}[$ and the sets

$$B_k^1 = \big\{x \in \Omega;\ \lambda\big(U(I_k, x) \cap A_{k+1}\big) \geqslant 2^{-n_k-k} \lambda\big(U(I_k, x)\big)\big\}.$$

The set $B_k^1$ depends only on the coordinates in $I_k$. It is the union of some of the sets of the type $U(x, I_k)$ and thus $\lambda(B_k^1 \cap A_{k+1}) \geqslant 2^{-n_k-k}\lambda(B_k^1)$ and, in particular, $\lambda(B_k^1) \leqslant 2^{n_k+k}\lambda(A_{k+1}) \leqslant 2^{-k}$. We further define

$$B_k^2 = \big\{x \in \Omega;\ \lambda\big(U(I_k, x) \cap A_k\big) \geqslant 2^{-n_{k-1}-k} \lambda\big(U(I_k, x)\big)\big\},$$

and, similarly $\lambda(B_k^2) \leqslant 2^{-k}$. We set $B_k = B_k^1 \cup B_k^2$ so that $\lambda(B_k) \leqslant 2^{-k-1}$. Thus $\sum_k \lambda(B_k) < \infty$ and the set $B_k$ depends only on the coordinates in $I_k$.

If it is the case where each $x \in \mathcal{F}$ belongs to infinitely many sets $B_k$, the proof is finished. So we may assume that this is not the case, and we fix $x \in \mathcal{F}$ and $k_0$ such that $x \notin B_k$ for $k \geqslant k_0$. We then define $C_k = C_k^1 \cup C_k^2$ where

$$C_k^1 = \big\{y \in \Omega;\ U(I_k, y) \cap U(I_{k+1}, x) \cap A_{k+1} \neq \emptyset\big\},$$
$$C_k^2 = \big\{y \in \Omega;\ U(I_k, y) \cap U(I_{k-1}, x) \cap A_k \neq \emptyset\big\}.$$

Since the set $A_{k+1}$ depends only on the coordinates $< n_{k+2}$, we have

$$y \in C_k^1 \quad \Rightarrow \quad \lambda\big(U(I_k, y) \cap U(I_{k+1}, x) \cap A_{k+1}\big) \geqslant 2^{-n_{k+2}} = 2^{-n_k}\lambda\big(U(I_k, y)\big)\lambda\big(U(I_{k+1}, x)\big),$$

so that summation over the disjoint sets of the type $U(I_k, y) \subset C_k^1$ yields

$$\lambda\big(C_k^1 \cap U(I_{k+1}, x) \cap A_{k+1}\big) \geqslant 2^{-n_k}\lambda\big(C_k^1\big)\lambda\big(U(I_{k+1}, x)\big).$$

For $k > k_0$, we have $x \notin B_{k+1}^2$ and thus $\lambda(U(I_{k+1}, x) \cap A_{k+1}) \leqslant 2^{-n_k-k}\lambda(U(I_{k+1}, x))$. Consequently, $\lambda(C_k^1) \leqslant 2^{-k}$. By a similar argument, we see that $\lambda(C_k^2) \leqslant 2^{-k}$. Thus if $C_k = C_k^1 \cup C_k^2$ we have $\sum_k \lambda(C_k) < \infty$.

To conclude the proof, we show that any $z \in \mathcal{F}$ belongs to infinitely many sets $C_k$. Consider $y \in \Omega$ given by $y_i = z_i$ if $i$ belong to an interval $I_k$ for $k$ even, and $y_i = x_i$ otherwise. Then $y \in \mathcal{F}$ because $x, z \in \mathcal{F}$ and $x \cap z \subset y$. Note also by construction that $y \in U(I_k, x)$ when $k$ is odd. Consider $q \geqslant k_0 + 1$ arbitrarily large. Then $y \in K_{n_q}^{n_q} = \Omega$ while $y \notin K^{n_q}$. Thus there is largest $p \geqslant q$ such that $y \in K_{n_p}^{n_q}$. Then $y \in K_{n_p}^{n_q} \setminus K_{n_{p+1}}^{n_q} \subset A_p$. Assume first that $p$ is odd. Then $y \in U(I_p, x)$, $y \in U(I_{p-1}, y)$, so that it is obvious that $y \in C_{p-1}^1 \subset C_{p-1}$. Assume next that $p$ is even. Then $p-1$ is odd, so that $y \in U(I_{p-1}, x)$, $y \in U(I_p, y)$ and it is now obvious that $y \in C_p^2 \subset C_p$. $\square$

## 3. Remarks on measurable filters

For $0 < p < 1$ let us now denote by $\lambda_p$ the product measure that gives weight $p$ to 1, so that $\lambda = \lambda_{1/2}$. The author proved in [2] that if a filter $\mathcal{F}$ satisfies $\lambda_p(\mathcal{F}) = 0$ for one $0 < p < 1$, then this is also the case for each $0 < p < 1$. Unfortunately, Theorem 1 does not make this result obvious.

Following an idea of T. Bartoszynski, for a number $0 < p < 1$, let us say that a filter $\mathcal{F}$ satisfies property $1_p$ if there exists a sequence $(I_k)$ of finite sets such that:

$$\sum_k p^{\operatorname{card} I_k} < \infty$$

and such that each element of $\mathcal{F}$ contains infinitely many sets $I_k$. (Here we do not require that the sets $I_k$ be disjoint.) Obviously, if $\mathcal{F}$ satisfies property $1_p$, then $\lambda_p(\mathcal{F}) = 0$, so that $\mathcal{F}$ is measurable. T. Bartoszynski's initial idea was that any measurable filter might have property $1_{1/2}$. Theorem 6 below shows that this is not true, but this concept nonetheless raises a number of natural problems, which might be connected to potentially difficult problems in combinatorics [3].

**Problem 2.** If a filter satisfies property $1_p$ for one $0 < p < 1$, does it satisfy property $1_p$ for each $0 < p < 1$?

The difficulty is that given sets $I_k$ which witness that $\mathcal{F}$ has property $1_{1/2}$, to prove property $1_p$ for $p > 1/2$ one has to find "much larger" sets than the sets $I_k$ (or maybe a very small subcollection of these sets) such that any element of the filter contains infinitely many of these.

There is a related notion which is more adapted to the change of value of $p$. Let us say that a filter satisfies property $2_p$ if for each finite set $I$ one can find a number $c_I \geqslant 0$ such that:

$$\sum_I c_I p^{\operatorname{card} I} < \infty$$

and such that for every element $x$ of $\mathcal{F}$ one has $\sum_{I \subset x} c_I = \infty$. Property $1_p$ is stronger than property $2_p$ as can be seen by taking $c_I = 1$ if $I$ is one of the sets $I_k$ and $c_I = 0$ otherwise.

**Proposition 3.** *If a filter has property $2_p$ for one $0 < p < 1$ it has this property for each $0 < p < 1$.*

**Proof.** Since property $2_p$ becomes stronger as $p$ increases, it suffices to prove that if a filter $\mathcal{F}$ has property $2_p$, then it has property $2_{\sqrt{p}}$. So, consider the numbers $c_I$ which witness that $\mathcal{F}$ has property $2_p$. If it happens that for each $x$ in $\mathcal{F}$ we have $\sum_{I \subset x} c_I p^{\operatorname{card} I/2} = \infty$, then, since the numbers $d_I = c_I p^{\operatorname{card} I/2}$ satisfy $\sum_I d_I p^{\operatorname{card} I/2} = \sum_I c_I p^{\operatorname{card} I} < \infty$, then $\mathcal{F}$ has property $2_{\sqrt{p}}$. Otherwise, there exists $x$ in $\mathcal{F}$ such that $\sum_{I \in x} c_I p^{\operatorname{card} I/2} < \infty$. Let us then define $d_I = c_I$ if $I \subset x$ and $d_I = 0$ otherwise. Then $\sum_I d_I p^{\operatorname{card} I/2} < \infty$ and for each $y$ in $\mathcal{F}$ we have $x \cap y \in \mathcal{F}$ so that:

$$\sum_{I \in y} d_I \geqslant \sum_{I \in x \cap y} c_I = \infty,$$

and thus $\mathcal{F}$ has property $2_{\sqrt{p}}$. □

**Problem 4.** If a filter has property $2_p$ for all $0 < p < 1$ does it have property $1_p$ for all $p$, or at least for $p$ small enough?

The author proved in [2] that the intersection of countably many non-measurable filters is non-measurable. This raises the following question.

**Problem 5.** If the intersection of countably many filters has property $2_p$, does one of them have property $2_p$?

**Theorem 6.** *Assuming Continuum Hypothesis, there exists a measurable filter which fails property $2_p$ for each $p$.*

Considering disjoint finite sets $J_{k,\ell}$, $k, \ell \geqslant 1$ with $\operatorname{card} J_{k,\ell} = k$, we can even arrange that every element $x$ of the filter satisfies $\lim_{k\to\infty} \min_{\ell \geqslant 1} \operatorname{card}(x \cap J_{k,\ell}) = \infty$. The proof is similar to that of Theorem 2.8 of [1]. The combinatorics can be taken care of by the following proposition.

**Proposition 7.** *Consider numbers $c_I$ with $\sum_I c_I p^{\operatorname{card} I} < \infty$. Consider a set $x$ with*

$$\lim_{k\to\infty} \min_{\ell \geqslant 1} \operatorname{card}(x \cap J_{k,\ell}) = \infty.$$

*Then there is a subset $y$ of $x$ such that $\lim_{k\to\infty} \min_{\ell \geqslant 1} \operatorname{card}(y \cap J_{k,\ell}) = \infty$ for which $\sum_{I \subset y} c_I < \infty$.*

To prove this we find as many disjoint sets of cardinality $\geqslant 1/p$ inside each set $x \cap I_{k,\ell}$, and we apply the following.

**Lemma 8.** *Consider numbers $c_I$ with $\sum_I c_I p^{\operatorname{card} I} < \infty$. Consider disjoint sets $J_k$ of $\mathbb{N}$, each of cardinality $\geqslant 1/p$. Then there is a set $y$ which meets all of the $J_k$ but for which $\sum_{I \subset y} c_I < \infty$.*

**Proof.** The collection of sets $J \subset \bigcup_k J_k$ which meet each set $J_k$ in exactly one point is endowed with a natural probability measure $\mathsf{P}$. Given any finite set $I$, one has $\mathsf{P}(I \subset J) \leqslant p^{\operatorname{card} J}$. (Actually this probability is zero unless $I \subset \bigcup_k J_k$ and $\operatorname{card}(I \cap J_k) \leqslant 1$ for each $k$.) Thus the expected value of $\sum_{I \subset J} c_I$ is finite. □

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