Partial Differential Equations/Mathematical Physics

# The Goursat problem for the Einstein-Vlasov system: (II) The evolution of initial data 

# Problème de Goursat pour les équations d'Einstein-Vlasov : (II) L'évolution des données initiales 

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#### Abstract

We solve, locally in time, the evolution problem associated with the Einstein-Vlasov (EV) system, the initial data being specified on two intersecting smooth null hypersurfaces. The proof of the obtained result relies heavily on a fixed point method deployed in appropriate weighted Sobolev spaces. The main tools of this method consist of adequate Sobolev inequalities and Moser estimates combined with energy inequalities for first-order and second-order linear hyperbolic partial differential equations.


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## R É S U M É

Nous résolvons, localement en temps, le problème de l'évolution associé au système Einstein-Vlasov (EV), les données initiales étant portées par deux hypersurfaces caractéristiques régulières sécantes. La preuve du résultat obtenu repose essentiellement sur une méthode de point fixe déployée dans un cadre approprié d'espaces de Sobolev à poids. Les principaux ingrédients de cette méthode sont constitués des inégalités de Sobolev et des estimations de Moser adéquates, combinées aux inégalités énergétiques pour les équations aux dérivées partielles hyperboliques linéaires du premier ordre et du second ordre.
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## 1. The Einstein-Vlasov (EV) system

### 1.1. The general form of the EV system

In relativistic kinetic theory, the evolution of a collisionless gas is governed by the EV system. The main geometric objects consist of a four-dimensional orientable $C^{\infty}$ manifold $\mathcal{M}$, endowed with a Lorentzian metric $\widehat{g}$ (of signature -+++ ). A particle of rest-mass $m$ is described by a trajectory $s \rightarrow(y(s), q(s))$ in the tangent bundle $T \mathcal{M}$ such that $\frac{\mathrm{d} y(s)}{\mathrm{d} s}=q(s)$, and at each point $y(s)$ the 4 -momentum $q(s)$ of the particle is future oriented and satisfies $\widehat{g}_{i j}(y(s)) q^{i}(s) q^{j}(s)=-m^{2}$. Throughout the Note, the subscript "," denotes partial derivatives, and Einstein convention on repeated indices is used. The range of indices is as follows: $i, j, \ldots=1, \ldots, 4 ; A, B, \ldots=2, \ldots, 4$. The EV system reads (see $[1,2,7]$ ):

[^0]\[

$$
\begin{equation*}
\widehat{S}_{i j} \equiv \widehat{R}_{i j}-\frac{1}{2} \widehat{R} \widehat{g}_{i j}=\widehat{T}_{i j}, \quad q^{i} \frac{\partial f}{\partial y^{i}}-\widehat{\Gamma}_{j k}^{i} q^{j} q^{k} \frac{\partial f}{\partial q^{i}}=0 \tag{1}
\end{equation*}
$$

\]

where $\widehat{g}_{i j}$ are the covariant components of the metric $\widehat{g}$; they constitute the unknowns for the Einstein equations. $\widehat{R}_{i j}$ are the covariant components of the Ricci tensor and $\widehat{R}$ is the scalar curvature of the metric $\widehat{g} . \widehat{\Gamma}_{i j}^{k}$ are the Christoffel symbols of the metric $\widehat{g}$. $f$ is the distribution function which constitutes the unknown for the Vlasov equation. $\widehat{T}_{i j}$ are the covariant components of the stress-energy tensor, which is the source of the gravitational field created by the particles. In contravariant components, the stress-energy tensor is defined by:

$$
\begin{equation*}
\widehat{T}^{i j}(y)=-\int_{F_{y}} f(y, q) q^{i} q^{j} \frac{|\widehat{g}|^{\frac{1}{2}}}{q_{1}} \mathrm{~d} q^{2} \wedge \mathrm{~d} q^{3} \wedge \mathrm{~d} q^{4} \tag{2}
\end{equation*}
$$

where $F_{y}=\left\{q=\left(q^{i}\right) \in T_{y} \mathcal{M}: \widehat{g}_{i j}(y) q^{i} q^{j}=-m^{2}, 0<q^{1}\right\},|\widehat{g}|$ is the modulus of the determinant of $\left(\widehat{g}_{i j}\right)$.

### 1.2. The reduced EV system with appropriate unknowns and variables

It is well known that the Einstein equations, as they stand, are not hyperbolic. However, in wave coordinates they take the form (see [2,4]) $\widehat{R}_{i j}^{h}=\widehat{T}_{i j}$, where $\widehat{R}_{i j}^{h} \equiv \widehat{R}_{i j}-\frac{1}{2}\left(\widehat{g}_{i k} \widehat{\Gamma}_{, j}^{k}+\widehat{g}_{j k} \widehat{\Gamma}_{i}^{k}\right)=-\frac{1}{2} \widehat{g}^{k m} \widehat{g}_{i j, m k}+Q_{i j}$. Here $\widehat{\Gamma}^{k}=\widehat{g}^{i j} \widehat{\Gamma}_{i j}^{k}$, and $Q_{i j}$ is a rational function depending on the metric components and their first order derivatives. So, the reduced EV system in the local coordinates $(y, q)$ has the following form:

$$
\begin{equation*}
-\frac{1}{2} \widehat{g}^{k m} \widehat{g}_{i j, m k}+Q_{i j}=\widehat{T}_{i j}, \quad q^{i} \frac{\partial f}{\partial y^{i}}+Q^{i} \frac{\partial f}{\partial q^{i}}=0 \tag{3}
\end{equation*}
$$

where $Q^{i}=-\widehat{\Gamma}_{j k}^{i} q^{j} q^{k}$. As a relativistic speed is bounded, we think that it is convenient to choose on the mass shell coordinates with bounded domain. In [10], following Choquet-Bruhat [2] and using convenient assumptions, we introduced new variable ( $x, v$ ) and a new unknown distribution function $\varphi$ such that the reduced EV system (4) takes the form:

$$
\begin{equation*}
\widetilde{R}_{i j}=T_{i j}, \quad H^{i} \frac{\partial \varphi}{\partial x^{i}}+L^{C} \frac{\partial \varphi}{\partial v^{C}}+F \varphi=0 \tag{4}
\end{equation*}
$$

where $\widetilde{R}_{i j} \equiv R_{i j}-\frac{1}{2}\left(g_{k i} \Gamma_{, j}^{k}+g_{k j} \Gamma_{, i}^{k}\right)=-\frac{1}{2} g^{k m} g_{i j, m k}+Q_{i j}$. Here $g_{i j}$ are the components of the unknown space-time metric in local coordinates ( $x^{i}$ ), with corresponding Ricci curvature $R_{i j}$, Christoffel symbols $\Gamma_{i j}^{k}$, and $\Gamma^{k}=g^{i j} \Gamma_{i j}^{k}$. The stress-energy tensor (2) is given in the local coordinates ( $x^{i}$ ) and the local parameters ( $v^{A}$ ) by (see Proposition 1.5 of [10]) $T^{i j}(x)=$ $\frac{1}{2 m^{2}} \int_{B} \varphi(x, v) v^{i j}|g|^{\frac{1}{2}}\left(-\widehat{g}^{11}\right)^{-\frac{3}{2}}|\widetilde{g}|^{-\frac{1}{2}} \mathrm{~d} v^{2} \wedge \mathrm{~d} v^{3} \wedge \mathrm{~d} v^{4}$, where $|\widetilde{g}|$ is the modulus of the determinant of $\left(\widehat{g}_{A B}\right), B$ is the unit open ball in $\mathbb{R}^{3}$. The explicit expressions of $H^{i}, L^{C}$ and $F$ are known (see Proposition 5 of [8]).

## 2. The evolution problem for the EV system

It is worth recalling, as clearly explained in [10], that the resolution of the evolution problem for the EV system (1) amounts to solving the initial value problem for the nonlinear system (4). To this end, we first define the functional framework for the studied problem.

### 2.1. Functional framework

Here we follow the notations and definitions of $[3,4,9]$. $L$ denotes a compact domain of $\mathbb{R}^{4}$ with a piecewise smooth boundary $\partial L$; $G^{1}$ and $G^{2}$ are two 3-dimensional surfaces defined by $G^{\omega}=\left\{x \in L: x^{\omega}=0\right\}, \omega=1$, 2 , where $x=\left(x^{a}\right)=$ $\left(x^{1}, \ldots, x^{4}\right)$ is the global canonical coordinates system of $\mathbb{R}^{4}$. Assume $G^{1} \cup G^{2} \subset \partial L$, set $\tau(x)=x^{1}+x^{2}$ and $T_{0}=\sup _{x \in L} \tau(x)$. For $t \in\left[0, T_{0}\right]$ and $s \in \mathbb{N}$, define the following point sets and weighted Sobolev spaces:

$$
\begin{aligned}
& L_{t}=\{x \in L: 0 \leqslant \tau(x) \leqslant t\}, \quad \Lambda_{t}=\{x \in L: \tau(x)=t\}, \quad G_{t}^{\omega}=\left\{x \in G^{\omega}: 0 \leqslant \tau(x) \leqslant t\right\}, \\
& \Gamma_{t}^{\omega}=\left\{x \in G^{\omega}: \tau(x)=t\right\}, \quad \Gamma=G^{1} \cap G^{2} ; \quad\|v\|_{H_{s}\left(S_{t}, R\right)}=t^{-\alpha}\left(\sum_{k=0}^{s} \int_{S_{t}}\left|D_{R}^{k} v\right|^{2} \mathrm{~d} S_{t}\right)^{\frac{1}{2}}, \\
& \|v\|_{H_{s}\left(S_{t}\right)} \equiv\|v\|_{H_{s}\left(S_{t}, S_{t}\right)}, \quad\|v\|_{E_{s}\left(S_{t}\right)}=\underset{\substack{\operatorname{ess} \sup \\
0 \leqslant \sigma \leqslant t}}{ }\|v\|_{H_{s}\left(\Sigma_{\sigma}, S_{t}\right)},
\end{aligned}
$$

where $\Sigma_{\sigma}=S_{t} \cap \Lambda_{\sigma}$. Here $\alpha=\frac{1}{2}$ if $S_{t} \in\left\{\Lambda_{t}, G_{t}^{\omega}\right\}, \alpha=1$ if $S_{t}=L_{t}, \alpha=0$ if $S_{t} \in\left\{\Gamma_{t}^{\omega}, \Gamma\right\}$. $R$ is a surface of $\mathbb{R}^{4}$ such that $S_{t} \subset R \subset L, D_{R}^{k}$ denote $k$-th order derivatives (in the sense of distributions) tangent to $R,\left|D_{R}^{k} v\right|$ is the norm of $D_{R}^{k} v$ w.r.t. the Kronecker metric $\delta^{a b}, \mathrm{~d} S_{t}$ is the volume element induced on $S_{t}$ by $\mathrm{d} x^{1} \ldots \mathrm{~d} x^{4}$.

Further norms (and corresponding spaces) are defined as follows:

$$
\begin{array}{ll}
\|v\|_{\mathbb{H}_{s}\left(G_{t}^{\omega}\right)}=\left[\sum_{k=0}^{s-1}\left(\left\|D_{\omega}^{k} v\right\|_{H_{2(s-k)-1}\left(G_{t}^{\omega}\right)}\right)^{2}\right]^{\frac{1}{2}}, \quad\|v\|_{\mathbb{H}_{s}\left(L_{t}\right)}=\left[\left(\|v\|_{H_{s}\left(L_{t}\right)}\right)^{2}+\sum_{\omega=1}^{2}\left(\|v\|_{\mathbb{H}_{s}\left(G_{t}^{\omega}\right)}\right)^{2}\right]^{\frac{1}{2}} \\
\|v\|_{\mathbb{E}_{s}\left(G_{t}^{\omega}\right)}=\left[\sum_{k=0}^{s-1}\left(\left\|D_{\omega}^{k} v\right\|_{E_{2(s-k)-1}\left(G_{t}^{\omega}\right)}\right)^{2}\right]^{\frac{1}{2}}, \quad\|v\|_{\mathbb{E}_{s}\left(L_{t}\right)}=\left[\left(\|v\|_{E_{s}\left(L_{t}\right)}\right)^{2}+\sum_{\omega=1}^{2}\left(\|v\|_{\mathbb{E}_{s}\left(G_{t}^{\omega}\right)}\right)^{2}\right]^{\frac{1}{2}}
\end{array}
$$

$C^{\infty}\left(L_{t}\right)$ denotes the space of restrictions to $L_{t}$ of functions which are $C^{\infty}$ on $\mathbb{R}^{4}$. From the Hilbertian structure of the usual Sobolev space $H^{s}\left(L_{t}\right)$, it is easy to see that $H_{s}\left(L_{t}\right)$ is a Hilbert space and $C^{\infty}\left(L_{t}\right)$ is dense in $H_{s}\left(L_{t}\right)$. $\mathbb{H}_{s}\left(L_{t}\right)$ denotes the completion of $C^{\infty}\left(L_{t}\right)$ with respect to the norm $\|v\|_{\mathbb{H}_{s}\left(L_{t}\right)}$. $E_{s}\left(M_{t}\right)$ and $\mathbb{E}_{s}\left(L_{t}\right)$ are Banach spaces. For $S_{t} \in\left\{L_{t}, \Lambda_{t}, G_{t}^{\omega}, \Gamma_{t}^{\omega}, L, \Gamma\right\}$, setting $\widehat{S_{t}}=S_{t} \times B$, where $B$ is the open unit ball in $\mathbb{R}^{3}$, one can define analogue norms on $\widehat{S_{t}}$ as above.

### 2.2. Solving the nonlinear system (4) in $\mathbb{E}_{S}\left(L_{T}\right) \times \mathbb{E}_{S}\left(\widehat{L_{T}}\right)$

The initial value problem for the nonlinear system (4) can be written as follows:

$$
\begin{cases}g^{a b}(x, u) D_{a} D_{b} u=f(x, u, D u, \varphi) \text { in } L_{T}, & u=\underset{\omega}{u} \text { on } G_{T}^{\omega},  \tag{5}\\ H^{\widehat{a}}(\widehat{x}, u, D u) D_{\widehat{a}} \varphi+F . \varphi=0 \text { in } \widehat{L_{T}}, & \varphi=\underset{\omega}{\varphi} \text { on } \widehat{G_{T}^{\omega}}\end{cases}
$$

where $x=\left(x^{a}\right), \widehat{x}=\left(x^{\widehat{a}}\right), u=\left(u^{I}\right)=\left(g_{i j}\right)$ is an unknown vector function (representing the space-time metric), $\varphi$ is the unknown distribution function, $f=\left(f^{l}\right)=\left(f_{i j}\right)$ is the vector function defined by $f_{i j}(u, D u, \varphi)=2\left[Q_{i j}(u, D u)-T_{i j}(\varphi)\right]$, $D_{a} u=\frac{\partial u^{I}}{\partial x^{a}}, D_{\widehat{a}} \varphi=\frac{\partial \varphi}{\partial x^{a}}$. The range of indices is as follows: $\omega=1,2 ; a=1, \ldots, 4 ; \widehat{a}=1, \ldots, 7 ; I=1, \ldots, 10$. We are now ready to state the main result of this Note (we refer the reader to $[3,4,9]$ for the concept of regular hyperbolicity used in assumption (iii)).

Theorem. Let $T>0$ be a real number, $s \geqslant 5$ an integer, $\omega \in\{1,2\}, \underset{\omega}{U}=(\underset{\omega}{u}, \underset{\omega}{\varphi})$ a given vector function such that:
(i) $\left.\underset{\omega}{u} \in E_{2 s-1}\left(G_{T}^{\omega}\right), \underset{\omega}{[\underset{\omega}{u}}\right]_{\Gamma} \in H_{2 s-1}(\Gamma), \underset{1}{u}=\underset{2}{u}$ on $\Gamma$,
(ii) $\varphi_{\omega}^{\varphi} \in E_{2 s-1}\left(\widehat{G_{T}^{\omega}}\right), \varphi_{1}^{\varphi}$ and $\varphi_{2}$ as well as all their existing derivatives vanish on $\widehat{\Gamma}$,
(iii) The metric $\underset{\omega}{u}$ is continuous and regularly hyperbolic on $G_{T}^{\omega}$,
(iv) $G_{T}^{\omega}$ is characteristic w.r.t. the metric $\underset{\omega}{u}$.

Then $\left.\exists T_{1} \in\right] 0, T\left[\right.$ such that the evolution problem (5) has a unique solution $(u, \varphi)$ in $\mathbb{E}_{s}\left(L_{T_{1}}\right) \times \mathbb{E}_{s}\left(\widehat{L_{T_{1}}}\right)$.
Strategy of the proof. The method used is based on the results of Dossa and Tadmon [3,4] concerning the resolution of the characteristic initial value problem for a class of quasilinear hyperbolic systems. Before developing the strategy of the proof, we introduce the spaces:

$$
\widetilde{\mathbb{E}}_{s}\left(L_{T}\right)=\left\{z=\left(g_{i j}\right) \in \mathbb{E}_{s}\left(L_{T}\right): z=\underset{\omega}{u} \text { on } G_{T}^{\omega}\right\}, \quad \widetilde{\mathbb{E}}_{s}\left(\widehat{L_{T}}\right)=\left\{\varphi \in \mathbb{E}_{S}\left(\widehat{L_{T}}\right): \varphi=\varphi_{\omega} \text { on } \widehat{G_{T}^{\omega}}\right\},
$$

and consider the following three mappings: $\beta: \widetilde{\mathbb{E}}_{S}\left(L_{T}\right) \rightarrow \widetilde{\mathbb{E}}_{\Sigma}\left(\widehat{L_{T}}\right)$, where $\varphi$ solves the first order linear Cauchy problem:

$$
\begin{equation*}
H^{\widehat{a}}(\widehat{x}, z, D z) D_{\widehat{a}} \varphi+F(\widehat{x}, z, D z) \varphi=0 \quad \text { in } \widehat{L_{T}}, \quad \varphi=\varphi_{\omega} \quad \text { on } \widehat{G_{T}^{\omega}} \tag{6}
\end{equation*}
$$

$\theta: \widetilde{\mathbb{E}}_{s}\left(L_{T}\right) \times \underset{(z, \varphi) \mapsto \theta(z, \varphi)=u}{ } \widetilde{\mathbb{E}}_{s}\left(\widehat{L_{T}}\right) \rightarrow \widetilde{\mathbb{E}}_{s}\left(L_{T}\right)$, where $u$ solves the second order linear Cauchy problem:

$$
\begin{equation*}
g^{a b}(z) D_{a} D_{b} u=f(z, D z, \varphi) \quad \text { in } L_{T}, \quad u=\underset{\omega}{u} \quad \text { on } G_{T}^{\omega} \tag{7}
\end{equation*}
$$

$\gamma: \widetilde{\mathbb{E}}_{S}\left(L_{T}\right) \rightarrow \widetilde{\mathbb{E}}_{s}\left(L_{T}\right) \times \gamma(z)=(z, \beta(z)) \leq \widetilde{\mathbb{E}}_{s}\left(\widehat{L_{T}}\right)$. We aim at proving that $\kappa=\theta \circ \gamma$ (the composition of mappings) is a contraction from a ball of $\widetilde{\mathbb{E}}_{S}\left(L_{T_{1}}\right)$ into itself, for some $T_{1}>0$. The strategy of the proof consists of three main steps.

First step: construction of an element of $\widetilde{\mathbb{E}}_{S}\left(L_{T}\right) \times \widetilde{\mathbb{E}}_{S}\left(\widehat{L_{T}}\right)$. This is done by considering the following linear Cauchy problem:

$$
\begin{cases}\gamma^{a b} D_{a b} z=0 \text { in } L_{T}, & z=\underset{\omega}{u} \text { on } G_{T}^{\omega}  \tag{8}\\ H^{\widehat{a}}(\widehat{x}) D_{\widehat{a}} \phi=0 \text { in } \widehat{L_{T}}, & \phi=\underset{\omega}{\varphi} \text { on } \widehat{G_{T}^{\omega}}\end{cases}
$$

where the non-vanishing $\gamma^{a b}$ are $\gamma^{12}=\gamma^{21}=-1, \gamma^{33}=\gamma^{44}=1$; $H^{\widehat{a}}=\left(\frac{1}{2}\left(1+v^{2}\right), \frac{1}{2}\left(1-v^{2}\right), v^{3}, v^{4}, 0,0,0\right)$. By using tools similar to those in [6], one can show that the linear problem (8) has a unique solution $Z=(z, \phi) \in \mathbb{E}_{S}\left(L_{T}\right) \times \mathbb{E}_{s}\left(\widehat{L_{T}}\right)$, for $5 \leqslant s \in \mathbb{N}$.

Second step: the mapping $\kappa$ is well defined. One uses appropriate Sobolev inequalities and Moser estimates as in $[3,4,9]$ to show, for $(z, \varphi) \in \widetilde{\mathbb{E}}_{s}\left(L_{T}\right) \times \widetilde{\mathbb{E}}_{s}\left(\widehat{L_{T}}\right)$, that $H(., z, D z), F(., z, D z) \in \mathbb{H}_{s}\left(\widehat{L_{T}}\right) ; g^{a b}(., z) \in \mathbb{H}_{s}\left(L_{T}\right) ; f(z, D z, \varphi) \in \mathbb{H}_{s-1}\left(L_{T}\right)$, for $5 \leqslant s \in \mathbb{N}$. Then one argues by using similar tools as in [6] to prove global existence and uniqueness for the linear problems (6) and (7).

Third step: the mapping $\kappa$ is a contraction from a ball of $\widetilde{\mathbb{E}}_{s}\left(L_{T_{1}}\right)$ into itself. Let $w \in \widetilde{\mathbb{E}}_{S}\left(L_{T}\right)$ and $u=\kappa(w)$. One derives the following inequality by using, repeatedly, suitable Sobolev inequalities and Moser estimates established in [3,4,9] (see for instance Theorems 4.2, 5.9 and 5.10 of [4]):

$$
\begin{equation*}
\|u\|_{\mathbb{E}_{s}\left(L_{t}\right)} \leqslant c_{1}\left[\sum_{\omega=1}^{2}\|u\|_{E_{2 s-1}\left(G_{t}^{\omega}\right)}+\sum_{\omega=1}^{2}\left\|\varphi_{\omega}\right\|_{E_{2 s-1}\left(\widehat{G}_{t}^{\omega}\right)}+t\left[1+\|w\|_{\mathbb{E}_{s}\left(L_{t}\right)}\right]^{2 s-3}\right], \quad t \in(0, T], \tag{9}
\end{equation*}
$$

where $c_{1}>0$ does not depend on $t$. Taking $\rho=\max \left\{\|z\|_{\mathbb{E}_{s}\left(L_{t}\right)}, 2 c_{1}\left(\sum_{\omega=1}^{2}\|\underset{\omega}{u}\|_{E_{2 s-1}\left(G_{t}^{\omega}\right)}+\sum_{\omega=1}^{2}\left\|\varphi_{\omega}\right\|_{E_{2 s-1}\left(\widehat{G_{t}^{\omega}}\right)}\right)\right\}$, it is easy to see from (9) that there exists $\delta \in(0, T]$ such that, for $t \in(0, \delta]$,

$$
\begin{equation*}
\|w\|_{\mathbb{E}_{s}\left(L_{t}\right)} \leqslant \rho \quad \Rightarrow \quad\|u\|_{\mathbb{E}_{s}\left(L_{t}\right)} \leqslant \rho \tag{10}
\end{equation*}
$$

Let now $w_{i} \in \widetilde{\mathbb{E}}_{s}\left(L_{t}\right), t \in(0, \delta]$, such that $\left\|w_{i}\right\|_{L_{t}, s} \leqslant \rho ; u_{i}=\kappa\left(w_{i}\right), i=1,2$. Appropriate Sobolev inequalities and Moser estimates (as in $[3,4,9]$ ) yield the following inequality for $t \in(0, \delta]$ :

$$
\begin{equation*}
\left\|u_{1}-u_{2}\right\|_{\mathbb{E}_{s-1}\left(L_{t}\right)} \leqslant c(t, \rho)\left\|w_{1}-w_{2}\right\|_{\mathbb{E}_{s-1}\left(L_{t}\right)} \tag{11}
\end{equation*}
$$

where $c(t, \rho)>0$ is a non-decreasing continuous function of its arguments, such that $\lim _{t \rightarrow 0} c(t, \rho)=0$. It then follows from (11) that there exists $T_{1} \in(0, \delta]$ such that:

$$
\begin{equation*}
\left\|u_{1}-u_{2}\right\|_{\mathbb{E}_{s-1}\left(L_{T_{1}}\right)}<\frac{1}{2}\left\|w_{1}-w_{2}\right\|_{\mathbb{E}_{s-1}\left(L_{T_{1}}\right)} . \tag{12}
\end{equation*}
$$

Now set $B_{\rho, T_{1}}=\left\{w \in \widetilde{\mathbb{E}}_{s}\left(L_{T_{1}}\right):\|w\|_{\mathbb{E}_{s}\left(L_{T_{1}}\right)} \leqslant \rho\right\}$. As pointed out in $[3,4,9], B_{\rho, T_{1}}$ (equipped with the distance defined by the norm $\|\cdot\|_{L_{t}, s-1}$ ) is a non-empty complete metric space. One deduces from (10) and (12) that $\kappa$ is a contraction from $B_{\rho, T_{1}}$ into itself. Therefore $\kappa$ has a unique fixed point $u \in B_{\rho, T_{1}}$. The uniqueness of $u$ in $\mathbb{E}_{s}\left(L_{T_{1}}\right)$ is a straightforward consequence of the Gronwall lemma and energy inequality for the second order linear $\underset{\sim}{\sim}$ Goursat problem. Thus $(u, \varphi)$, where $\varphi=\beta(u)$, is the unique solution of the nonlinear evolution problem (5) in $\widetilde{\mathbb{E}}_{s}\left(L_{T_{1}}\right) \times \widetilde{\mathbb{E}}_{s}\left(\widehat{L_{T_{1}}}\right)$.

Remark. This Note complements a recent paper [10] where we solved the constraints problem for the EV system. Moreover it is, to a great extent, the characteristic counterpart of previous work [2,5] devoted to the standard Cauchy problem (with initial data prescribed on a spacelike hypersurface) for the EV system.

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## References

[1] H. Andreasson, Living Rev. Relativ. 14 (2011) 4, lrr-2011-4.
[2] Y. Choquet-Bruhat, Ann. Inst. Fourier 21 (3) (1971) 181-201.
[3] M. Dossa, C. Tadmon, C. R. Acad. Sci. Paris, Ser. I 348 (2010) 35-39.
[4] M. Dossa, C. Tadmon, Appl. Math. Res. Express 2010 (2) (2010) 154-231.
[5] P.B. Mucha, J. Appl. Anal. 4 (1) (1998) 111-127.
[6] H. Müller zum Hagen, Ann. Inst. Henri Poincaré, Phys. Théor. 53 (1990) 159-216.
[7] A.D. Rendall, arXiv:gr-qc/0208082v1, 2002.
[8] C. Tadmon, arXiv:1109.6844v1 [gr-qc], 2011.
[9] C. Tadmon, Editions universitaires européennes, ISBN 978-3-8381-8631-3, 2012, 132 pp .
[10] C. Tadmon, The Goursat problem for the Einstein-Vlasov system: (I) The initial data constraints, C. R. Math. Acad. Sci. Soc. R. Can. (2013), in press.


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