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Mathematical Problems in Mechanics

Expression of Dirichlet boundary conditions in terms of the strain tensor in linearized elasticity

Expression de conditions aux limites de Dirichlet en fonction du tenseur linéarisé des déformations

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ABSTRACT

In a previous work, it was shown how the linearized strain tensor field $\mathbf{e} := \frac{1}{2}(\nabla \boldsymbol{u}^T + \nabla \boldsymbol{u}) \in \mathbb{L}^2(\Omega)$ can be considered as the sole unknown in the Neumann problem of linearized elasticity posed over a domain $\Omega \subset \mathbb{R}^3$, instead of the displacement vector field $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ in the usual approach. The purpose of this Note is to show that the same approach applies as well to the Dirichlet–Neumann problem. To this end, we show how the boundary condition $\boldsymbol{u} = \mathbf{0}$ on a portion Γ_0 of the boundary of Ω can be recast, again as boundary conditions on Γ_0, but this time expressed only in terms of the new unknown $\mathbf{e} \in \mathbb{L}^2(\Omega)$.

RÉSUMÉ

Dans un travail antérieur, on a montré comment le champ $\mathbf{e} := \frac{1}{2}(\nabla \boldsymbol{u}^T + \nabla \boldsymbol{u}) \in \mathbb{L}^2(\Omega)$ des tenseurs linéarisés des déformations peut être considéré comme la seule inconnue dans le problème de Neumann pour l'élasticité linéarisée posé sur un domaine $\Omega \subset \mathbb{R}^3$, au lieu du champ $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ des déplacements dans l'approche habituelle. L'objet de cette Note est de montrer que la même approche s'applique aussi bien au problème de Dirichlet–Neumann. À cette fin, nous montrons comment la condition aux limites $\boldsymbol{u} = \mathbf{0}$ sur une portion Γ_0 de la frontière de Ω peut être ré-écrite, à nouveau sous forme de conditions aux limites sur Γ_0, mais exprimées cette fois uniquement en fonction de la nouvelle inconnue $\mathbf{e} \in \mathbb{L}^2(\Omega)$.

1. Preliminaries

Greek indices, resp. Latin indices, range over the set $\{1,2\}$, resp. $\{1,2,3\}$. The summation convention with respect to repeated indices is used in conjunction with these rules. The notations $|\mathbf{a}|$, $\mathbf{a} \wedge \boldsymbol{b}$, $\mathbf{a} \otimes \boldsymbol{b}$, and $\mathbf{a} \cdot \boldsymbol{b}$ respectively denote the Euclidean norm, the exterior product, the dyadic product, and the inner product of vectors $\mathbf{a}, \boldsymbol{b} \in \mathbb{R}^3$.

The notation $\mathbb{S}^m$, resp. $\mathbb{A}^m$, designates the space of all symmetric, resp. antisymmetric, tensors of order m. The inner product of two $m \times m$ tensors $\mathbf{e}$ and $\boldsymbol{\tau}$ is denoted and defined by $\mathbf{e} : \boldsymbol{\tau} = \operatorname{tr}(\mathbf{e}^T \boldsymbol{\tau})$. Given a normed vector space X, the notation $\mathcal{L}^2_{\text{sym}}(X \times X)$ designates the space of all continuous symmetric bilinear forms defined on the product $X \times X$.

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Let $\Omega \subset \mathbb{R}^3$ be a connected, bounded, open set whose boundary $\partial\Omega$ is of class $\mathcal{C}^4$. This means that there exist a finite number N of open sets $\omega^k \subset \mathbb{R}^2$ and of mappings $\boldsymbol{\theta}^k \in \mathcal{C}^4(\omega^k; \mathbb{R}^3)$, $k = 1, 2, \ldots, N$, such that $\partial\Omega = \bigcup_{k=1}^N \boldsymbol{\theta}^k(\omega^k)$. It also implies that there exists $\varepsilon > 0$ such that the mappings $\boldsymbol{\Theta}^k \in \mathcal{C}^3(U^k; \mathbb{R}^3)$, defined by:

$$\boldsymbol{\Theta}^k(y, y_3) := \boldsymbol{\theta}^k(y) + y_3\mathbf{a}_3^k(y) \quad \text{for all } (y, y_3) \in U^k := \omega^k \times (-\varepsilon, \varepsilon),$$

where $\mathbf{a}_3^k$ denotes the unit inner normal vector field along the portion $\boldsymbol{\theta}^k(\omega^k)$ of the boundary of Ω, are $\mathcal{C}^3$-diffeomorphisms onto their image (cf. [2, Theorem 4.1-1]). Thus the mappings $\{\boldsymbol{\Theta}^k;\ 1 \leqslant k \leqslant N\}$ form an atlas of local charts for the open set $\Omega_\varepsilon := \{x \in \Omega; \operatorname{dist}(x, \partial\Omega) < \varepsilon\} \subset \mathbb{R}^3$, while the mappings $\{\boldsymbol{\theta}^k; 1 \leqslant k \leqslant N\}$ form an atlas of local charts for the surface $\Gamma = \partial\Omega \subset \mathbb{R}^3$. When no confusion should arise, we will drop the explicit dependence on k for notational brevity.

A generic point in ω is denoted $y = (y_\alpha)$ and a generic point in $U = \omega \times (-\varepsilon, \varepsilon)$ is denoted (y, y_3). Partial derivatives with respect to y_i are denoted ∂_i. The vectors $\mathbf{a}_\alpha(y) := \partial_\alpha\boldsymbol{\theta}(y)$ form a basis in the tangent space at $\boldsymbol{\theta}(y)$ to the surface $\Gamma := \partial\Omega \subset \mathbb{R}^3$ and the vectors $\mathbf{g}_i(y, y_3) := \partial_i\boldsymbol{\Theta}(y, y_3)$ form a basis in the tangent space at $\boldsymbol{\Theta}(y, y_3)$ to the open set $\boldsymbol{\Theta}(U) \subset \Omega_\varepsilon \subset \mathbb{R}^3$. Note that:

$$\mathbf{g}_\alpha(y, y_3) = \mathbf{a}_\alpha(y) + y_3\partial_\alpha\mathbf{a}_3(y) \quad \text{and} \quad \mathbf{g}_3(y, y_3) = \mathbf{a}_3(y).$$

By exchanging if necessary the coordinates y_1 and y_2, we may always assume that:

$$\mathbf{a}_3(y) = \frac{\mathbf{a}_1(y) \wedge \mathbf{a}_2(y)}{|\mathbf{a}_1(y) \wedge \mathbf{a}_2(y)|}.$$

The vectors $\mathbf{a}^\alpha(y)$ in the tangent space at $\boldsymbol{\theta}(y)$ to Γ and $\mathbf{g}^i(y, y_3)$ in the tangent space at $\boldsymbol{\Theta}(y, y_3)$ are defined by:

$$\mathbf{a}^\alpha(y) \cdot \mathbf{a}_\beta(y) = \delta_\beta^\alpha \quad \text{and} \quad \mathbf{g}^i(y, y_3) \cdot \mathbf{g}_j(y, y_3) = \delta_j^i,$$

the area element on Γ is $\mathrm{d}\Gamma := \sqrt{a}dy$, where $a := |\mathbf{a}_1 \wedge \mathbf{a}_2|$, and the Christoffel symbols $C_{\alpha\beta}^\sigma$ and Γ_{ij}^k, respectively induced by the immersions $\boldsymbol{\theta}$ and $\boldsymbol{\Theta}$, are defined by:

$$C_{\alpha\beta}^\sigma := \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \mathbf{a}^\sigma \quad \text{and} \quad \Gamma_{ij}^k := \partial_{ij}\boldsymbol{\Theta} \cdot \mathbf{g}^k.$$

A point in Ω will be specified either by its Cartesian coordinates $x = (x_i)$ with respect to a given orthonormal basis $\hat{\mathbf{e}}^i$ in $\mathbb{R}^3$, or, when $x \in \Omega_\varepsilon \subset \Omega$, by its curvilinear coordinates (y, y_3) corresponding to a local chart $\boldsymbol{\Theta}$; thus $x = \boldsymbol{\Theta}(y, y_3)$ in such a local chart.

Vector fields, resp. tensor fields, on Ω will be expanded at each $x = \boldsymbol{\Theta}(y, y_3) \in \Omega_\varepsilon$ over the contravariant bases $\mathbf{g}^i(y, y_3)$, resp. $(\mathbf{g}^i \otimes \mathbf{g}^j)(y, y_3)$. Covariant derivatives with respect to the local chart $\boldsymbol{\Theta}$ are defined as usual, and denoted $u_{i\|j}$, $u_{i\|jk}$, $e_{ij\|k}$, etc.

Let Γ_0 be a connected and relatively open subset of the boundary Γ of Ω. Since Γ is a manifold of class $\mathcal{C}^4$, so is Γ_0. It follows that functions, vector fields, and tensor fields, of class $\mathcal{C}^m$, $m = 0, 1, 2$, can be defined on Γ_0. The Lebesgue and Sobolev spaces on Γ_0 and their norms are then defined as in, e.g., Aubin [1].

We also let $\mathcal{C}_c^m(\Gamma_0)$ denote the space of all functions $f : \Gamma_0 \to \mathbb{R}$ of class $\mathcal{C}^m$ with compact support contained in Γ_0. Then the Sobolev space $H_0^m(\Gamma_0)$ is defined as the completion of the space $\mathcal{C}_c^m(\Gamma_0)$ with respect to the norm $\|\cdot\|_{H^m(\Gamma_0)}$. Its dual space is denoted $H^{-m}(\Gamma_0)$.

Spaces of vector fields, resp. symmetric tensor fields, with values in $\mathbb{R}^3$, resp. in $\mathbb{S}^3$, are defined by using a given Cartesian basis $\{\hat{\mathbf{e}}_i, 1 \leqslant i \leqslant 3\}$ in $\mathbb{R}^3$, resp. the basis $\{\frac{1}{2}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j + \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_i), 1 \leqslant i, j \leqslant 3\}$ in $\mathbb{S}^3$. They will be denoted by bold letters and by capital Roman letters, respectively.

Complete proofs and complements will be found in [5].

2. Linearized change of metric and curvature tensors on $\partial\Omega$ associated with a linearized strain tensor in $\mathbb{C}^1(\overline{\Omega})$

Given any displacement field $\boldsymbol{u} \in \boldsymbol{\mathcal{C}}^2(\overline{\Omega})$, the restriction $\boldsymbol{\zeta} := \boldsymbol{u}|_{\overline{\Gamma}_0} \in \boldsymbol{\mathcal{C}}^2(\overline{\Gamma}_0)$ is a displacement field of the surface $\overline{\Gamma}_0 \subset \mathbb{R}^3$. The linearized change of metric and change of curvature tensor fields induced by $\boldsymbol{\zeta}$ are then respectively defined in each local chart by:

$$\begin{aligned} \boldsymbol{\gamma}(\boldsymbol{\zeta}) &= \gamma_{\alpha\beta}(\boldsymbol{\zeta})\mathbf{a}^\alpha \otimes \mathbf{a}^\beta, \quad \text{where } \gamma_{\alpha\beta}(\boldsymbol{\zeta}) := \frac{1}{2}(\partial_\alpha\boldsymbol{\zeta} \cdot \mathbf{a}_\beta + \partial_\beta\boldsymbol{\zeta} \cdot \mathbf{a}_\alpha), \\ \boldsymbol{\rho}(\boldsymbol{\zeta}) &= \rho_{\alpha\beta}(\boldsymbol{\zeta})\mathbf{a}^\alpha \otimes \mathbf{a}^\beta, \quad \text{where } \rho_{\alpha\beta}(\boldsymbol{\zeta}) := \big(\partial_{\alpha\beta}\boldsymbol{\zeta} - C_{\alpha\beta}^\sigma\partial_\sigma\boldsymbol{\zeta}\big) \cdot \mathbf{a}_3, \end{aligned} \tag{1}$$

where for convenience the *same* notation $\boldsymbol{\zeta}$ denotes *either* the vector field $\boldsymbol{\zeta} : \overline{\Gamma}_0 \to \mathbb{R}^3$ *or* the vector field $\boldsymbol{\zeta} := \boldsymbol{\zeta} \circ \boldsymbol{\theta} : \omega \to \mathbb{R}^3$ in a local chart $\boldsymbol{\theta} : \omega \to \overline{\Gamma}_0$ of $\overline{\Gamma}_0$.

Let $T_x\Gamma_0 \subset \mathbb{R}^3$ denote the tangent space at each point x of the surface Γ_0. Given any matrix field $\mathbf{e} \in \mathbb{C}^1(\overline{\Omega})$, let the tensor fields:

$$\boldsymbol{\gamma}^\sharp(\mathbf{e}) : x \in \Gamma_0 \to \big(\boldsymbol{\gamma}^\sharp(\mathbf{e})\big)(x) \in \mathcal{L}^2_{\mathrm{sym}}(T_x\Gamma_0 \times T_x\Gamma_0),$$
$$\boldsymbol{\rho}^\sharp(\mathbf{e}) : x \in \Gamma_0 \to \big(\boldsymbol{\rho}^\sharp(\mathbf{e})\big)(x) \in \mathcal{L}^2_{\mathrm{sym}}(T_x\Gamma_0 \times T_x\Gamma_0),$$

be defined in a local chart $\boldsymbol{\theta} : \omega \to \Gamma_0$ by:

$$\boldsymbol{\gamma}^\sharp(\mathbf{e}) = \gamma^\sharp_{\alpha\beta}(\mathbf{e})\mathbf{a}^\alpha \otimes \mathbf{a}^\beta, \quad \text{where } \gamma^\sharp_{\alpha\beta}(\mathbf{e}) := e_{\alpha\beta}|_{\omega\times\{0\}},$$
$$\boldsymbol{\rho}^\sharp(\mathbf{e}) = \rho^\sharp_{\alpha\beta}(\mathbf{e})\mathbf{a}^\alpha \otimes \mathbf{a}^\beta, \quad \text{where } \rho^\sharp_{\alpha\beta}(\mathbf{e}) := \big(e_{\alpha 3\|\beta} + e_{\beta 3\|\alpha} - e_{\alpha\beta\|3} + \Gamma^3_{\alpha\beta} e_{33}\big)\big|_{\omega\times\{0\}}, \tag{2}$$

and the functions e_{ij} are defined by $\mathbf{e}(x) = (e_{ij}\boldsymbol{g}^i \otimes \mathbf{g}^j)(y, y_3)$ for all $x = \boldsymbol{\Theta}(y, y_3)$.

The following theorem shows that the tensors fields $\boldsymbol{\gamma}(\boldsymbol{\zeta})$ and $\boldsymbol{\rho}(\boldsymbol{\zeta})$, which are defined in terms of the trace on Γ_0 of the *displacement field* $\boldsymbol{u} \in \boldsymbol{\mathcal{C}}^2(\overline{\Omega})$, can be in fact expressed in terms of the traces on Γ_0 of the *linearized strain tensor field*:

$$\mathbf{e} = \nabla_s \boldsymbol{u} := \frac{1}{2}\big(\nabla \boldsymbol{u}^T + \nabla \boldsymbol{u}\big) \in \mathbb{C}^1(\overline{\Omega}).$$

Theorem 1. *Let $\boldsymbol{u} \in \boldsymbol{\mathcal{C}}^2(\overline{\Omega})$, let $\mathbf{e} = \nabla_s \boldsymbol{u}$, and let $\boldsymbol{\zeta} := \boldsymbol{u}|_{\overline{\Gamma}_0}$. Then*:

$$\boldsymbol{\gamma}(\boldsymbol{\zeta}) = \boldsymbol{\gamma}^\sharp(\mathbf{e}) \quad \text{and} \quad \boldsymbol{\rho}(\boldsymbol{\zeta}) = \boldsymbol{\rho}^\sharp(\mathbf{e}) \quad \text{on } \overline{\Gamma}_0,$$

where the tensor fields appearing in these equalities are defined in (1) *and* (2).

Sketch of proof. Proving the above equalities amounts to proving the equalities:

$$\gamma^\sharp_{\alpha\beta}(\mathbf{e}) = \gamma_{\alpha\beta}(\boldsymbol{\zeta}) \quad \text{and} \quad \rho^\sharp_{\alpha\beta}(\mathbf{e}) = \rho_{\alpha\beta}(\boldsymbol{\zeta}) \quad \text{in } \omega,$$

in any local chart $\boldsymbol{\theta} : \omega \to \overline{\Gamma}_0$ of the surface $\overline{\Gamma}_0$. These equalities follow from direct computations. □

Let:

$$\operatorname{Im} \nabla_s := \big\{\nabla_s \boldsymbol{u};\ \boldsymbol{u} \in \boldsymbol{\mathcal{C}}^2(\overline{\Omega})\big\} \subset \mathbb{C}^1(\overline{\Omega}),$$

and let the linear operators:

$$\boldsymbol{\gamma}^\sharp : \mathbf{e} \in \operatorname{Im}\nabla_s \mapsto \boldsymbol{\gamma}^\sharp(\mathbf{e}) \in \mathbb{C}^1(\overline{\Gamma}_0) \quad \text{and} \quad \boldsymbol{\rho}^\sharp : \mathbf{e} \in \operatorname{Im}\nabla_s \mapsto \boldsymbol{\rho}^\sharp(\mathbf{e}) \in \mathbb{C}^0(\overline{\Gamma}_0)$$

be defined by the relations (2). The next theorem shows that these operators are continuous with respect to appropriate “weak” norms.

Theorem 2.

(a) *There exists a constant C such that*:

$$\big\|\boldsymbol{\gamma}^\sharp(\mathbf{e})\big\|_{\mathbb{H}^{-1}(\Gamma_0)} + \big\|\boldsymbol{\rho}^\sharp(\mathbf{e})\big\|_{\mathbb{H}^{-2}(\Gamma_0)} \leqslant C \inf_{\boldsymbol{r}\in\boldsymbol{R}(\Omega)} \big\|(\boldsymbol{u}+\boldsymbol{r})|_{\Gamma_0}\big\|_{\boldsymbol{L}^2(\Gamma_0)} \quad \text{for all } \mathbf{e} = \nabla_s\boldsymbol{u},\ \boldsymbol{u} \in \boldsymbol{\mathcal{C}}^2(\overline{\Omega}),$$

where

$$\boldsymbol{R}(\Omega) := \big\{\boldsymbol{r} : \Omega \to \mathbb{R}^3;\ \text{there exist } \mathbf{a} \in \mathbb{R}^3 \text{ and } \boldsymbol{B} \in \mathbb{A}^3 \text{ such that } \boldsymbol{r}(x) = \mathbf{a} + \boldsymbol{B}x,\ x \in \Omega\big\}.$$

(b) *There exists a constant C such that*:

$$\big\|\boldsymbol{\gamma}^\sharp(\mathbf{e})\big\|_{\mathbb{H}^{-1}(\Gamma_0)} + \big\|\boldsymbol{\rho}^\sharp(\mathbf{e})\big\|_{\mathbb{H}^{-2}(\Gamma_0)} \leqslant C\|\mathbf{e}\|_{\mathbb{L}^2(\Omega)} \quad \text{for all } \mathbf{e} = \nabla_s\boldsymbol{u},\ \boldsymbol{u} \in \boldsymbol{\mathcal{C}}^2(\overline{\Omega}).$$

Sketch of proof. Let $\mathbf{e} \in \operatorname{Im}\nabla_s$ and let $\boldsymbol{u} \in \boldsymbol{\mathcal{C}}^2(\overline{\Omega})$ be any vector field that satisfies $\mathbf{e} = \nabla_s\boldsymbol{u}$. Since the spaces $\mathbb{C}^1_c(\Gamma_0)$ and $\mathbb{C}^2_c(\Gamma_0)$ are respectively dense in the spaces $\mathbb{H}^1_0(\Gamma_0)$ and $\mathbb{H}^2_0(\Gamma_0)$, we have:

$$\big\|\boldsymbol{\gamma}^\sharp(\mathbf{e})\big\|_{\mathbb{H}^{-1}(\Gamma_0)} = \sup_{\boldsymbol{\tau}\in\mathbb{C}^1_c(\Gamma_0)} \frac{|\int_{\Gamma_0} \boldsymbol{\gamma}^\sharp(\mathbf{e}) : \boldsymbol{\tau}\, \mathrm{d}\Gamma|}{\|\boldsymbol{\tau}\|_{\mathbb{H}^1(\Gamma_0)}} \quad \text{and} \quad \big\|\boldsymbol{\rho}^\sharp(\mathbf{e})\big\|_{\mathbb{H}^{-2}(\Gamma_0)} = \sup_{\boldsymbol{\tau}\in\mathbb{C}^2_c(\Gamma_0)} \frac{|\int_{\Gamma_0} \boldsymbol{\rho}^\sharp(\mathbf{e}) : \boldsymbol{\tau}\, \mathrm{d}\Gamma|}{\|\boldsymbol{\tau}\|_{\mathbb{H}^2(\Gamma_0)}}.$$

The basic idea of the proof then relies on a careful re-writing of the numerators found in the above expressions. Combined with a partition of unity associated with a covering $\Gamma_0 \subset \bigcup_{k=1}^N \boldsymbol{\theta}^k(\omega^k)$, this re-writing shows that there exist constants C_1 and C_2 such that:

$$\left|\int_{\Gamma_0} \boldsymbol{\gamma}^\sharp(\mathbf{e}) : \boldsymbol{\tau}\,\mathrm{d}\Gamma\right| \leqslant C_1 \|\boldsymbol{\zeta}\|_{\boldsymbol{L}^2(\Gamma_0)} \|\boldsymbol{\tau}\|_{\mathbb{H}^1(\Gamma_0)} \quad \text{for all } \boldsymbol{\tau} \in \mathbb{C}_c^1(\Gamma_0),$$

and

$$\left|\int_{\Gamma_0} \boldsymbol{\rho}^\sharp(\mathbf{e}) : \boldsymbol{\tau}\,\mathrm{d}\Gamma\right| \leqslant C_2 \|\boldsymbol{\zeta}\|_{\boldsymbol{L}^2(\Gamma_0)} \|\boldsymbol{\tau}\|_{\mathbb{H}^2(\Gamma_0)} \quad \text{for all } \boldsymbol{\tau} \in \mathbb{C}_c^2(\Gamma_0),$$

where $\boldsymbol{\zeta} := \boldsymbol{u}|_{\Gamma_0}$, so that:

$$\left\|\boldsymbol{\gamma}^\sharp(\mathbf{e})\right\|_{\mathbb{H}^{-1}(\Gamma_0)} \leqslant C_1 \|\boldsymbol{\zeta}\|_{\boldsymbol{L}^2(\Gamma_0)} = C_1 \|\boldsymbol{u}|_{\Gamma_0}\|_{\boldsymbol{L}^2(\Gamma_0)} \tag{3}$$

and

$$\left\|\boldsymbol{\rho}^\sharp(\mathbf{e})\right\|_{\mathbb{H}^{-2}(\Gamma_0)} \leqslant C_2 \|\boldsymbol{\zeta}\|_{\boldsymbol{L}^2(\Gamma_0)} = C_2 \|\boldsymbol{u}|_{\Gamma_0}\|_{\boldsymbol{L}^2(\Gamma_0)}. \tag{4}$$

Since inequalities (3) and (4) hold for all vector fields $\boldsymbol{u} \in \boldsymbol{\mathcal{C}}^2(\overline{\Omega})$ that satisfy $\mathbf{e} = \nabla_s \boldsymbol{u}$, and since $\nabla_s \boldsymbol{r} = \mathbf{0}$ for all $\boldsymbol{r} \in \boldsymbol{R}(\Omega)$, there exists a constant C_3 such that:

$$\left\|\boldsymbol{\gamma}^\sharp(\mathbf{e})\right\|_{\mathbb{H}^{-1}(\Gamma_0)} + \left\|\boldsymbol{\rho}^\sharp(\mathbf{e})\right\|_{\mathbb{H}^{-2}(\Gamma_0)} \leqslant C_3 \inf_{\boldsymbol{r} \in \boldsymbol{R}(\Omega)} \left\|(\boldsymbol{u} + \boldsymbol{r})|_{\Gamma_0}\right\|_{\boldsymbol{L}^2(\Gamma_0)}.$$

Furthermore, the continuity of the trace operator from $\boldsymbol{H}^1(\Omega)$ into $\boldsymbol{L}^2(\Gamma_0)$ shows that there exists a constant C_4 such that:

$$\left\|\boldsymbol{\gamma}^\sharp(\mathbf{e})\right\|_{\mathbb{H}^{-1}(\Gamma_0)} + \left\|\boldsymbol{\rho}^\sharp(\mathbf{e})\right\|_{\mathbb{H}^{-2}(\Gamma_0)} \leqslant C_4 \inf_{\boldsymbol{r} \in \boldsymbol{R}(\Omega)} \|\boldsymbol{u} + \boldsymbol{r}\|_{\boldsymbol{H}^1(\Omega)}.$$

Finally, the classical Korn's inequality shows that there exists a constant C_5 such that:

$$\left\|\boldsymbol{\gamma}^\sharp(\mathbf{e})\right\|_{\mathbb{H}^{-1}(\Gamma_0)} + \left\|\boldsymbol{\rho}^\sharp(\mathbf{e})\right\|_{\mathbb{H}^{-2}(\Gamma_0)} \leqslant C_5 \|\nabla_s \boldsymbol{u}\|_{\mathbb{L}^2(\Omega)} = C_5 \|\mathbf{e}\|_{\mathbb{L}^2(\Omega)}. \qquad \square$$

3. Linearized change of metric and curvature tensors on $\partial\Omega$ associated with a linearized strain tensor in $\mathbb{L}^2(\Omega)$

The classical Korn's inequality shows that the closure of the space $\operatorname{Im}\nabla_s$ in the space $\mathbb{L}^2(\Omega)$ is the space:

$$\overline{\operatorname{Im}\nabla_s} = \left\{\nabla_s \boldsymbol{u};\ \boldsymbol{u} \in \boldsymbol{H}^1(\Omega)\right\} \subset \mathbb{L}^2(\Omega).$$

The next theorem shows that the definition of the tensor fields $\boldsymbol{\gamma}^\sharp(\mathbf{e})$ and $\boldsymbol{\rho}^\sharp(\mathbf{e})$ given in Section 2 for fields $\mathbf{e} \in \operatorname{Im}\nabla_s \subset \mathbb{C}^1(\overline{\Omega})$ can be extended in a natural way to linearized strain vector fields $\mathbf{e}$ that belong to the closed subspace $\overline{\operatorname{Im}\nabla_s}$ of $\mathbb{L}^2(\Omega)$.

Theorem 3. *Let the linear operators*:

$$\boldsymbol{\gamma}^\sharp : \mathbf{e} \in \operatorname{Im}\nabla_s \mapsto \boldsymbol{\gamma}^\sharp(\mathbf{e}) \in \mathbb{C}^1(\overline{\Gamma}_0) \quad \textit{and} \quad \boldsymbol{\rho}^\sharp : \mathbf{e} \in \operatorname{Im}\nabla_s \mapsto \boldsymbol{\rho}^\sharp(\mathbf{e}) \in \mathbb{C}^0(\overline{\Gamma}_0)$$

be defined by (2).

(a) *There exist continuous linear operators*:

$$\overline{\boldsymbol{\gamma}}^\sharp : \mathbf{e} \in \overline{\operatorname{Im}\nabla_s} \mapsto \overline{\boldsymbol{\gamma}}^\sharp(\mathbf{e}) \in \mathbb{H}^{-1}(\Gamma_0) \quad \textit{and} \quad \overline{\boldsymbol{\rho}}^\sharp : \mathbf{e} \in \overline{\operatorname{Im}\nabla_s} \mapsto \overline{\boldsymbol{\rho}}^\sharp(\mathbf{e}) \in \mathbb{H}^{-2}(\Gamma_0)$$

such that:

$$\overline{\boldsymbol{\gamma}}^\sharp(\mathbf{e}) = \boldsymbol{\gamma}^\sharp(\mathbf{e}) \quad \textit{and} \quad \overline{\boldsymbol{\rho}}^\sharp(\mathbf{e}) = \boldsymbol{\rho}^\sharp(\mathbf{e}) \quad \textit{for all } \mathbf{e} \in \operatorname{Im}\nabla_s,$$

and there exists a constant C_0 such that:

$$\left\|\overline{\boldsymbol{\gamma}}^\sharp(\mathbf{e})\right\|_{\mathbb{H}^{-1}(\Gamma_0)} + \left\|\overline{\boldsymbol{\rho}}^\sharp(\mathbf{e})\right\|_{\mathbb{H}^{-2}(\Gamma_0)} \leqslant C_0 \|\mathbf{e}\|_{\mathbb{L}^2(\Omega)} \quad \textit{for all } \mathbf{e} = \nabla_s \boldsymbol{u},\ \boldsymbol{u} \in \boldsymbol{H}^1(\Omega).$$

(b) *There exists a constant C_1 such that*:

$$\left\|\overline{\boldsymbol{\gamma}}^\sharp(\mathbf{e})\right\|_{\mathbb{H}^{-1}(\Gamma_0)} + \left\|\overline{\boldsymbol{\rho}}^\sharp(\mathbf{e})\right\|_{\mathbb{H}^{-2}(\Gamma_0)} \leqslant C_1 \inf_{\boldsymbol{r} \in \boldsymbol{R}(\Omega)} \left\|(\boldsymbol{u} + \boldsymbol{r})|_{\Gamma_0}\right\|_{\boldsymbol{L}^2(\Gamma_0)} \quad \textit{for all } \mathbf{e} = \nabla_s \boldsymbol{u},\ \boldsymbol{u} \in \boldsymbol{H}^1(\Omega).$$

Proof. It suffices to combine Theorem 2 with the classical theorem about the extension of densely defined continuous linear operators with values in a Banach space. $\square$

4. A Korn inequality on a surface in Sobolev spaces with negative exponents

In the proof of Theorem 5 in the next section, we will need a “weak” variant of the *Korn inequality on a surface* (Theorem 4), the difference with the *classical Korn inequality on a surface* (see, e.g., [2, Theorem 4.3-5]) being that it is now expressed in terms of *negative* Sobolev norms. If $\boldsymbol{\zeta}$ is only in the space $\boldsymbol{L}^2(\Gamma_0)$, the corresponding linearized change of metric, and change of curvature, tensor fields $\boldsymbol{\gamma}(\boldsymbol{\zeta}) \in \mathbb{H}^{-1}(\Gamma_0)$, and $\boldsymbol{\rho}(\boldsymbol{\zeta}) \in \mathbb{H}^{-2}(\Gamma_0)$, are defined in a local chart $\boldsymbol{\theta}$ by the formulas (1).

Theorem 4. *Let the sets Ω and Γ_0 satisfy the assumptions of Section 1 and let the space $\boldsymbol{R}(\Gamma_0)$ be defined by*:

$$\boldsymbol{R}(\Gamma_0) := \left\{\boldsymbol{r} : \Gamma_0 \to \mathbb{R}^3;\ \textit{there exist } \mathbf{a} \in \mathbb{R}^3 \textit{ and } \boldsymbol{B} \in \mathbb{A}^3 \textit{ such that } \boldsymbol{r}(x) = \mathbf{a} + \boldsymbol{B}x,\ x \in \Gamma_0\right\}.$$

Then there exists a constant C such that:

$$\inf_{\boldsymbol{r}\in\boldsymbol{R}(\Gamma_0)} \|\boldsymbol{\zeta} + \boldsymbol{r}\|_{\boldsymbol{L}^2(\Gamma_0)} \leqslant C\left(\left\|\boldsymbol{\gamma}(\boldsymbol{\zeta})\right\|_{\mathbb{H}^{-1}(\Gamma_0)} + \left\|\boldsymbol{\rho}(\boldsymbol{\zeta})\right\|_{\mathbb{H}^{-2}(\Gamma_0)}\right) \quad \textit{for all } \boldsymbol{\zeta} \in \boldsymbol{L}^2(\Gamma_0).$$

Sketch of proof. (i) It suffices first to prove that there exists a constant C such that:

$$\|\boldsymbol{\zeta}\|_{\boldsymbol{L}^2(\Gamma_0)} \leqslant C\left(\|\boldsymbol{\zeta}\|_{\boldsymbol{H}^{-1}(\Gamma_0)} + \left\|\boldsymbol{\gamma}(\boldsymbol{\zeta})\right\|_{\mathbb{H}^{-1}(\Gamma_0)} + \left\|\boldsymbol{\rho}(\boldsymbol{\zeta})\right\|_{\mathbb{H}^{-2}(\Gamma_0)}\right) \tag{5}$$

for all $\boldsymbol{\zeta} \in \boldsymbol{L}^2(\Gamma_0)$, then to establish the following *weak form of the infinitesimal rigid displacement lemma on a surface*: If a displacement field $\boldsymbol{\zeta} \in \boldsymbol{L}^2(\Gamma_0)$ satisfies $\boldsymbol{\gamma}(\boldsymbol{\zeta}) = \mathbf{0}$ in $\mathbb{H}^{-1}(\Gamma_0)$ and $\boldsymbol{\rho}(\boldsymbol{\zeta}) = \mathbf{0}$ in $\mathbb{H}^{-2}(\Gamma_0)$, then there exist a vector $\mathbf{a} \in \mathbb{R}^3$ and an antisymmetric matrix $\boldsymbol{B} \in \mathbb{A}^3$ such that $\boldsymbol{\zeta}(x) = \mathbf{a} + \boldsymbol{B}x$ for $\mathrm{d}\Gamma$-almost all $x \in \Gamma_0$.

(ii) *Proof of inequality* (5). It suffices to prove (5) only for $\boldsymbol{\zeta} \in \boldsymbol{\mathcal{C}}^2(\overline{\Gamma}_0)$. To this end, a crucial use is made of the relation:

$$2\partial_{\alpha\beta}\zeta_\sigma = \partial_\alpha(\partial_\beta\zeta_\sigma + \partial_\sigma\zeta_\beta) + \partial_\beta(\partial_\alpha\zeta_\sigma + \partial_\sigma\zeta_\alpha) - \partial_\sigma(\partial_\beta\zeta_\alpha + \partial_\alpha\zeta_\beta),$$

and of a crucial inequality due to Nečas [7] (see also Theorem 6.14-1 in [3]), which shows that there exists a constant C independent of $\boldsymbol{\zeta}$ such that:

$$\|\boldsymbol{\zeta}\|_{\boldsymbol{L}^2(\omega)} \leqslant C\left(\|\boldsymbol{\zeta}\|_{\boldsymbol{H}^{-1}(\omega)} + \sum_\alpha \|\partial_\alpha\boldsymbol{\zeta}\|_{\boldsymbol{H}^{-1}(\omega)}\right) \quad \text{for all } \boldsymbol{\zeta} \in \boldsymbol{L}^2(\omega).$$

(iii) Proving the *infinitesimal rigid displacement lemma* hinges on a careful adaptation of an argument due to P.G. Ciarlet and S. Mardare [6, Lemma 2] to vector fields $\boldsymbol{\zeta}$ that are only in $\boldsymbol{L}^2(\Gamma_0)$. $\square$

5. An intrinsic formulation of the boundary conditions

Using Theorems 3 and 4, we now show how *a homogeneous Dirichlet boundary condition imposed on the displacement field appearing in the displacement–traction problem of linearized elasticity can be replaced by a homogeneous boundary condition imposed on the linearized strain tensor field*:

Theorem 5. *Let the sets Ω and Γ_0 satisfy the assumptions of Section 1. Given a vector field $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, let*:

$$\mathbf{e} = \nabla_s\boldsymbol{u} \in \mathbb{L}^2(\Omega).$$

(a) *If $\boldsymbol{u} = \mathbf{0}$ on Γ_0, then $\overline{\boldsymbol{\gamma}}^\sharp(\mathbf{e}) = \mathbf{0}$ in $\mathbb{H}^{-1}(\Gamma_0)$ and $\overline{\boldsymbol{\rho}}^\sharp(\mathbf{e}) = \mathbf{0}$ in $\mathbb{H}^{-2}(\Gamma_0)$, where the tensor fields $\overline{\boldsymbol{\gamma}}^\sharp(\mathbf{e})$ and $\overline{\boldsymbol{\rho}}^\sharp(\mathbf{e})$ are those defined in Theorem 3.*

(b) *If $\overline{\boldsymbol{\gamma}}^\sharp(\mathbf{e}) = \mathbf{0}$ in $\mathbb{H}^{-1}(\Gamma_0)$ and $\overline{\boldsymbol{\rho}}^\sharp(\mathbf{e}) = \mathbf{0}$ in $\mathbb{H}^{-2}(\Gamma_0)$, then there exists a unique infinitesimal rigid displacement $\boldsymbol{r} \in \boldsymbol{R}(\Omega)$ such that $(\boldsymbol{u} + \boldsymbol{r}) = \mathbf{0}$ on Γ_0.*

Sketch of proof. (a) Assume that $\boldsymbol{u} = \mathbf{0}$ on Γ_0. The second inequality of Theorem 3 shows that:

$$\left\|\overline{\boldsymbol{\gamma}}^\sharp(\mathbf{e})\right\|_{\mathbb{H}^{-1}(\Gamma_0)} + \left\|\overline{\boldsymbol{\rho}}^\sharp(\mathbf{e})\right\|_{\mathbb{H}^{-2}(\Gamma_0)} \leqslant C_1\|\boldsymbol{u}|_{\Gamma_0}\|_{\boldsymbol{L}^2(\Gamma_0)}.$$

Since $\boldsymbol{u}$ vanishes on Γ_0, it then follows that $\overline{\boldsymbol{\gamma}}^\sharp(\mathbf{e}) = \mathbf{0}$ in $\mathbb{H}^{-1}(\Gamma_0)$ and $\overline{\boldsymbol{\rho}}^\sharp(\mathbf{e}) = \mathbf{0}$ in $\mathbb{H}^{-2}(\Gamma_0)$.

(b) Assume that $\overline{\boldsymbol{\gamma}}^\sharp(\mathbf{e}) = \mathbf{0}$ in $\mathbb{H}^{-1}(\Gamma_0)$ and that $\overline{\boldsymbol{\rho}}^\sharp(\mathbf{e}) = \mathbf{0}$ in $\mathbb{H}^{-2}(\Gamma_0)$. Since the space $\boldsymbol{\mathcal{C}}^2(\overline{\Omega})$ is dense in $\boldsymbol{H}^1(\Omega)$, there exists a sequence $(\boldsymbol{u}_n)$ in $\boldsymbol{\mathcal{C}}^2(\overline{\Omega})$ such that $\boldsymbol{u}_n \to \boldsymbol{u}$ in $\boldsymbol{H}^1(\Omega)$ as $n \to \infty$. Therefore,

$$\nabla_s\boldsymbol{u}_n \to \nabla_s\boldsymbol{u} = \mathbf{e} \quad \text{in } \mathbb{L}^2(\Omega) \text{ as } n \to \infty.$$

By Theorem 3, this implies that:

$$\overline{\boldsymbol{\gamma}}^\sharp(\nabla_s\boldsymbol{u}_n) \to \overline{\boldsymbol{\gamma}}^\sharp(\mathbf{e}) = \mathbf{0} \quad \text{in } \mathbb{H}^{-1}(\Gamma_0) \quad \text{and} \quad \overline{\boldsymbol{\rho}}^\sharp(\nabla_s\boldsymbol{u}_n) \to \overline{\boldsymbol{\rho}}^\sharp(\mathbf{e}) = \mathbf{0} \quad \text{in } \mathbb{H}^{-2}(\Gamma_0) \text{ as } n \to \infty.$$

Let $\boldsymbol{\zeta} := \boldsymbol{u}|_{\Gamma_0}$ and $\boldsymbol{\zeta}_n := \boldsymbol{u}_n|_{\Gamma_0}$. Since $\boldsymbol{u}_n \in \mathcal{C}^2(\overline{\Omega})$, Theorems 1 and 3 together show that:

$$\overline{\boldsymbol{\gamma}}^\sharp(\nabla_s \boldsymbol{u}_n) = \boldsymbol{\gamma}^\sharp(\nabla_s \boldsymbol{u}_n) = \boldsymbol{\gamma}(\boldsymbol{\zeta}_n) \quad \text{and} \quad \overline{\boldsymbol{\rho}}^\sharp(\nabla_s \boldsymbol{u}_n) = \boldsymbol{\rho}^\sharp(\nabla_s \boldsymbol{u}_n) = \boldsymbol{\rho}(\boldsymbol{\zeta}_n).$$

Hence the previous convergences become:

$$\boldsymbol{\gamma}(\boldsymbol{\zeta}_n) \to \mathbf{0} \quad \text{in } \mathbb{H}^{-1}(\Gamma_0) \quad \text{and} \quad \boldsymbol{\rho}(\boldsymbol{\zeta}_n) \to \mathbf{0} \quad \text{in } \mathbb{H}^{-2}(\Gamma_0) \text{ as } n \to \infty.$$

Combined with the Korn inequality established in Theorem 4, these convergences imply that:

$$\inf_{\boldsymbol{r} \in \boldsymbol{R}(\Gamma_0)} \|\boldsymbol{\zeta}_n + \boldsymbol{r}\|_{\boldsymbol{L}^2(\Gamma_0)} \to 0 \quad \text{as } n \to \infty.$$

There thus exists a sequence $(\boldsymbol{r}_n)$ in the space $\boldsymbol{R}(\Gamma_0)$ such that:

$$\boldsymbol{\zeta}_n + \boldsymbol{r}_n \to \mathbf{0} \quad \text{in } \boldsymbol{L}^2(\Gamma_0) \text{ as } n \to \infty.$$

The space $\boldsymbol{R}(\Gamma_0)$ being finite-dimensional, the sequence $(\boldsymbol{r}_n)$ possesses a convergent subsequence. Let $\boldsymbol{r}$ denote the limit of this subsequence; we then have $\boldsymbol{r} \in \boldsymbol{R}(\Gamma_0)$ and $\boldsymbol{\zeta} + \boldsymbol{r} = \mathbf{0}$ in $\boldsymbol{L}^2(\Gamma_0)$. Hence the trace on Γ_0 of the vector field $(\boldsymbol{u} + \boldsymbol{r}) \in \boldsymbol{H}^1(\Omega)$ vanishes.

If $\tilde{\boldsymbol{r}} \in \boldsymbol{R}(\Omega)$ is such that the trace on Γ_0 of the vector field $(\boldsymbol{u} + \boldsymbol{r}) \in \boldsymbol{H}^1(\Omega)$ also vanishes, then $(\tilde{\boldsymbol{r}} - \boldsymbol{r})|_{\Gamma_0} = \mathbf{0}$. This implies that $(\tilde{\boldsymbol{r}} - \boldsymbol{r}) = \mathbf{0}$ in Ω. □

We refer to the extended article [5] for applications of the results presented in this Note. There it will be shown in particular how *the Dirichlet–Neumann boundary value problem of three-dimensional linearized elasticity can be completely recast as a boundary value problem with the tensor field* $\mathbf{e} = \nabla_s \boldsymbol{u}$ *as the sole unknown*. Such a result thus complements the approach of [4], which was restricted to the *pure Neumann problem* of the three-dimensional linearized elasticity.

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References

[1] T. Aubin, Some Nonlinear Problems in Riemannian Geometry, Springer, Berlin, 2010.
[2] P.G. Ciarlet, An Introduction to Differential Geometry with Applications to Elasticity, Springer, Dordrecht, 2005.
[3] P.G. Ciarlet, Linear and Nonlinear Functional Analysis with Applications, SIAM, Providence, 2013.
[4] P.G. Ciarlet, P. Ciarlet Jr., Another approach to linearized elasticity and a new proof of Korn's inequality, Math. Models Methods Appl. Sci. 15 (2005) 259–271.
[5] P.G. Ciarlet, C. Mardare, Intrinsic formulation of the displacement–traction problem in linearized elasticity, in preparation.
[6] P.G. Ciarlet, S. Mardare, On Korn's inequalities in curvilinear coordinates, Math. Models Methods Appl. Sci. 11 (2001) 1379–1391.
[7] J. Nečas, Equations aux Dérivées Partielles, Presses de l'université de Montréal, 1966.