Algebraic Geometry

Stable bundles as Frobenius morphism direct image

Faisceaux stables en tant qu’images directes par le morphisme de Frobenius

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Abstract

Let \( X \) be a smooth projective curve of genus \( g \geq 2 \) over an algebraically closed field \( k \) of characteristic \( p > 0 \), and let \( F : X \to X_1 \) be the relative Frobenius morphism. We show that a vector bundle \( E \) on \( X_1 \) is the direct image under \( F \) of some stable bundle on \( X \) if and only if the instability of \( F^*E \) is equal to \((p-1)(2g-2)\).

1. Introduction

Let \( X \) be a smooth projective curve of genus \( g \geq 2 \) defined over an algebraically closed field \( k \) of characteristic \( p > 0 \). The absolute Frobenius morphism \( F_X : X \to X \) is induced by \( \mathcal{O}_X \to \mathcal{O}_X, f \mapsto f^p \). Let \( F : X \to X_1 := X \times_k k \) denote the relative Frobenius morphism over \( k \). One of the themes is to study its action on the geometric objects on \( X \). Recall that a vector bundle \( E \) on a smooth projective curve is called semi-stable (resp. stable) if \( \mu(E') \leq \mu(E) \) (resp. \( \mu(E') < \mu(E) \)) for any nontrivial proper subbundle \( E' \subset E \), where \( \mu(E) \) is the slope of \( E \). It is known that \( F_* \) preserves the stability of vector bundles (cf. [5]), but \( F^* \) does not preserve the semi-stability of vector bundles (cf. [1] for example).

Semi-stable bundles are basic constituents of vector bundles in the sense that any bundle \( E \) admits a unique filtration:

\[
\text{HN}_n(E) : 0 = \text{HN}_0(E) \subset \text{HN}_1(E) \subset \cdots \subset \text{HN}_n(E) = E,
\]

which is the so-called Harder–Narasimhan filtration, such that:

1. \( \text{gr}_i^{\text{HN}}(E) := \text{HN}_i(E)/\text{HN}_{i-1}(E) \) \( (1 \leq i \leq \ell) \) are semi-stable;
2. \( \mu(\text{gr}_1^{\text{HN}}(E)) > \mu(\text{gr}_2^{\text{HN}}(E)) > \cdots > \mu(\text{gr}_\ell^{\text{HN}}(E)) \).

The rational number \( I(E) := \mu(\text{gr}_1^{\text{HN}}(E)) - \mu(\text{gr}_\ell^{\text{HN}}(E)) \), which measures how far a vector bundle is from being semi-stable, is called the instability of \( E \). It is clear that \( E \) is semi-stable if and only if \( I(E) = 0 \).
Given a semi-stable bundle $E$ on $X_1$, then $F^*E$ may not be semi-stable, so it is natural to consider the instability $I(F^*E)$. In [4, Theorem 3.1], the author proves $I(F^*E) \leq (\ell - 1)(2g - 2)$, where $\ell$ is the length of Harder–Narasimhan filtration of $F^*E$. If $E = F_*W$ where $W$ is a stable bundle on $X$, we know, by Sun’s theorem [5, Theorem 2.2], that $E$ is stable, the length of Harder–Narasimhan filtration of $F^*E$ is $p$ and $I(F^*E) = (p - 1)(2g - 2)$. Thus $I(F^*E) = (p - 1)(2g - 2)$ is a necessary condition for $E$ to be a direct image under Frobenius. In this short note, we show the following theorem:

**Theorem 1.** Let $E$ be a stable vector bundle on $X$. Then the following statements are equivalent:

1. There exists a stable bundle $W$ such that $E = F_*W$;
2. $I(F^*E) = (p - 1)(2g - 2)$.

The case $\text{rk} E = p$ was proved in [3]. Our observation is that the arguments in [3] together with Sun’s theorem imply the general case.

**2. Proof of the theorem**

Let $X$ be a smooth projective curve over an algebraically closed field $k$ with char($k$) = $p > 0$. The absolute Frobenius morphism $F_X : X \to X$ is induced by the following homomorphism:

\[ O_X \to O_X, \quad f \mapsto f^p. \]

Let $F : X \to X_1 := X \times_k k$ denote the relative Frobenius morphism over $k$ that satisfies the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{F} & X_1 \\
\downarrow & & \downarrow \\
\text{Spec}(k) & \xrightarrow{F_k} & \text{Spec}(k).
\end{array}
\]

For a vector bundle $E$ on $X$, the slope of $E$ is defined as

\[ \mu(E) := \frac{\deg E}{\text{rk} E} \]

where $\text{rk} E$ (resp. $\deg E$) denotes the rank (resp. degree) of $E$. Then:

**Definition 1.** A vector bundle $E$ on $X$ is called semi-stable (resp. stable) if for any nontrivial proper subbundle $E' \subset E$, we have

\[ \mu(E') \leq (\text{resp.} <) \mu(E). \]

**Theorem 2 (Harder–Narasimhan filtration).** For any vector bundle $E$, there is a unique filtration:

\[ HN_*(E) : \quad 0 = HN_0(E) \subset HN_1(E) \subset \cdots \subset HN_\ell(E) = E, \]

which is called Harder–Narasimhan filtration, such that:

1. $\text{gr}_i^{\text{HN}}(E) := HN_i(E)/HN_{i-1}(E) (1 \leq i \leq \ell)$ are semi-stable;
2. $\mu(\text{gr}_1^{\text{HN}}(E)) > \mu(\text{gr}_2^{\text{HN}}(E)) > \cdots > \mu(\text{gr}_\ell^{\text{HN}}(E))$.

By using this unique filtration of $E$, an invariant $I(E)$ of $E$, which is called the instability of $E$ was introduced (see [5] and [4]). It is a rational number and measures how far is $E$ from being semi-stable.

**Definition 2.** Let $\mu_{\text{max}}(E) = \mu(\text{gr}_1^{\text{HN}}(E)), \mu_{\text{min}}(E) = \mu(\text{gr}_\ell^{\text{HN}}(E))$. Then the instability of $E$ is defined to be

\[ I(E) := \mu_{\text{max}}(E) - \mu_{\text{min}}(E). \]

It is easy to see that a vector bundle $E$ is semi-stable if and only if $I(E) = 0$. For any semi-stable bundle $E$, let

\[ HN_*(F^*E) : \quad 0 = HN_0(F^*E) \subset HN_1(F^*E) \subset \cdots \subset HN_\ell(F^*E) = F^*E \]

be the Harder–Narasimhan filtration of $F^*E$. Then we have the following lemma, which is implicit in [3].
**Lemma 1.** For any semi-stable bundle $E$, we have
\[
\begin{align*}
\mu_{\max}(F^*E) &\leq p \cdot \mu(E) + (p - 1)(g - 1); \\
\mu_{\min}(F^*E) &\geq p \cdot \mu(E) - (p - 1)(g - 1),
\end{align*}
\]
and if $I(F^*E) = \mu_{\max}(F^*E) - \mu_{\min}(F^*E) = (p - 1)(2g - 2)$. Then
\[
\begin{align*}
\mu_{\max}(F^*E) &= p \cdot \mu(E) + (p - 1)(g - 1); \\
\mu_{\min}(F^*E) &= p \cdot \mu(E) - (p - 1)(g - 1).
\end{align*}
\]

Now we prove our theorem by using this Lemma 1, the canonical filtration on the vector bundle $V = F^*F_\ast W$ and Sun’s theorem on the stability of Frobenius’ direct images.

**Proof of Theorem 1.** (1) $\Rightarrow$ (2). In [2, Section 5.3], there is a canonical filtration on the vector bundle $V = F^*F_\ast W$:
\[
0 = V_0 \subset V_1 \subset \cdots \subset V_{\ell - 1} \subset V_\ell \subset \cdots \subset V_{p - 1} \subset V_p = V,
\]
which is indeed the Harder–Narasimhan filtration on $V$, and satisfies
\[
V_\ell/V_{\ell - 1} \cong (V_{\ell + 1}/V_\ell) \otimes \Omega_K^1
\]
for $1 \leq \ell \leq p - 1$, and $V_p/V_{p - 1} \cong W$. So $\mu(V_{p}/V_{p - 1}) = \mu(W)$, $\mu(V_0/V_1) = \mu(W) + (p - 1)(2g - 2)$, and now the result is clear.

(2) $\Rightarrow$ (1). Since $I(F^*E) = (p - 1)(2g - 2)$, we have $\mu_{\max}(F^*E) = p \cdot \mu(E) + (p - 1)(g - 1)$, $\mu_{\min}(F^*E) = p \cdot \mu(E) - (p - 1)(g - 1)$ by Lemma 1. We consider the surjection:
\[
F^*E \rightarrow \text{gr}^\text{HN}_{\ell}(F^*E).
\]
The bundle $\text{gr}^\text{HN}_{\ell}(F^*E)$ is semi-stable of slope $\mu_{\min}(F^*E)$. Replacing $\text{gr}^\text{HN}_{\ell}(F^*E)$ by a stable graded piece $W$ in the Jordan–Hölder filtration of $\text{gr}^\text{HN}_{\ell}(F^*E)$, we have a surjection:
\[
F^*E \rightarrow W,
\]
where $W$ is a stable bundle of slope $\mu(W) = \mu_{\min}(F^*E) = p \cdot \mu(E) - (p - 1)(g - 1)$. By adjunction, we have a nontrivial morphism:
\[
\psi : E \rightarrow F_\ast W.
\]
By Sun’s theorem (cf. [5, Theorem 2.2]), we know that $F_\ast W$ is a stable bundle of slope:
\[
\mu(F_\ast W) = \frac{\mu(W)}{p} + \frac{(p - 1)(g - 1)}{p} = \mu(E).
\]
Thus $\psi$ induce an isomorphism:
\[
E \cong F_\ast W. \quad \square
\]

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**References**