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Functional Analysis

Inequality between unitary orbits

Inégalités entre orbites unitaires

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ABSTRACT

For bounded self-adjoint operators A and B we write $A \overset{u}{\leqslant} B$ if there is a unitary U such that $A \leqslant U^*BU$. In [7], Kosaki (1992) has shown that $A \leqslant B \Rightarrow \exp(A) \overset{u}{\leqslant} \exp(B)$. In this note, we extend this; especially, we show that for a function $f(t) = \sum_{i=1}^n c_i t^{a_i} e^{b_i t}$ with positive coefficients a_i, b_i and c_i, $0 \leqslant A \overset{u}{\leqslant} B \Rightarrow f(A) \overset{u}{\leqslant} f(B)$. We then apply this to a positive linear map and get a similar inequality.

RÉSUMÉ

Pour deux opérateurs autoadjoints bornés A et B, nous écrirons $A \overset{u}{\leqslant} B$ s'il existe un opérateur unitaire U tel que $A \leqslant U^*BU$. Kosaki (1992) a montré dans [7] que $A \leqslant B \Rightarrow \exp(A) \overset{u}{\leqslant} \exp(B)$. Cette note étend ce résultat. En particulier nous montrons que pour les fonctions du type $f(t) = \sum_{i=1}^n c_i t^{a_i} e^{b_i t}$ avec des coefficients a_i, b_i, c_i positifs, on a $0 \leqslant A \overset{u}{\leqslant} B \Rightarrow f(A) \overset{u}{\leqslant} f(B)$. Ceci permet d'obtenir des inégalités similaires pour les applications linéaires positives unitales.

1. Introduction

Let $B(\mathfrak{H})$ be the algebra of all bounded operators on a separable Hilbert space $\mathfrak{H}$ and $B_h(\mathfrak{H})$ the subset of all self-adjoint operators. For $A, B \in B_h(\mathfrak{H})$, we write $A \leqslant B$ if $(A\mathbf{x}, \mathbf{x}) \leqslant (B\mathbf{x}, \mathbf{x})$ for every $\mathbf{x} \in \mathfrak{H}$. Let $\{E_t\}$, $\{F_t\}$ be the spectral families for $A, B \in B_h(\mathfrak{H})$. Then $A \leqslant B$ clearly implies $\dim E_t\mathfrak{H} \geqslant \dim F_t\mathfrak{H}$. We write $A \preccurlyeq B$ if $E_t \geqslant F_t$ for every t, and this order is called *spectral order*. In [10] it was shown that if $A, B \geqslant 0$, then $A \preccurlyeq B$ if and only if $A^a \leqslant B^a$ for every $a > 0$. In [11,12] this concept was extended to (unbounded) self-adjoint operators A and B bounded from below and it was shown that $A \preccurlyeq B$ if and only if $e^{-tB} \leqslant e^{-tA}$ for every $t > 0$. Let $0 \leqslant A \leqslant B$, and let $B - A$ be of rank 1. Then $A \preccurlyeq B$ if $A^a \leqslant B^a$ for some $a > 1$ [13].

X and Y in $B(\mathfrak{H})$ are said to be *unitarily equivalent* or *unitarily similar* and denoted by $X \overset{u}{\simeq} Y$ if there is a unitary U such that $Y = U^*XU$. $O(X) := \{Y \in B(\mathfrak{H}):\ X \overset{u}{\simeq} Y\}$ is called the *unitary orbit* of X. For A and B in $B_h(\mathfrak{H})$ we write $A \overset{u}{\leqslant} B$ if there is a unitary U such that $A \leqslant U^*BU$. Likewise, we write $A \overset{u}{\preccurlyeq} B$ if there is a unitary U such that $A \preccurlyeq U^*BU$. It is evident that $A \overset{u}{\leqslant} B$ (or $A \overset{u}{\preccurlyeq} B$) if and only if there is $B' \in O(B)$ such that $A \leqslant B'$ (or $A \preccurlyeq B'$).

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To illuminate the importance of the concept $\overset{u}{\leqslant}$ in studying self-adjoint operators, we consider the case of $\dim \mathfrak{H} = n$; in this case we use conventional symbol H_n instead of $B_h(\mathfrak{H})$. Let $\lambda_1(A) \geqslant \cdots \geqslant \lambda_n(A)$ be eigenvalues of $A \in H_n$, where each eigenvalue is repeated as much as its multiplicity. Then $A \leqslant B$ implies $\lambda_i(A) \leqslant \lambda_i(B)$ $(i = 1, 2, \ldots, n)$ (e.g., see [2]). We therefore obtain:

(i) $A \overset{u}{\leqslant} B$ and $B \overset{u}{\leqslant} A$ if and only if $A \overset{u}{\simeq} B$.
(ii) If $A \overset{u}{\leqslant} B$, then $A \overset{u}{\preccurlyeq} B$.

The purpose of this paper is to study these facts for A and B in $B_h(\mathfrak{H})$. In [8] it was mentioned that if $0 \leqslant A \preccurlyeq U^*AU$, then $A = U^*AU$; but we will give a counterexample. We do not know if (ii) remains true for A and B in $B_h(\mathfrak{H})$. However, Kosaki [7], in virtue of an inequality by Ando [1], has shown the relevant inequality:

$$A \leqslant B \quad \Longrightarrow \quad \exp(A) \overset{u}{\leqslant} \exp(B). \tag{1}$$

We will extend this in a different way, and then deal with a positive linear map.

2. Inequalities

We first consider (i) for A and B in $B_h(\mathfrak{H})$. It is evident that $A \overset{u}{\simeq} B$ implies $A \overset{u}{\leqslant} B$ and $B \overset{u}{\leqslant} A$. But the converse implication does not hold (e.g., see [9]). We now show that even stronger condition $A \overset{u}{\preccurlyeq} B$, $B \overset{u}{\preccurlyeq} A$ does not necessarily imply $A \overset{u}{\simeq} B$ by giving a counterexample.

Example 1. Let A and B be diagonal operators on $\ell^2(\mathbf{Z})$ defined by

$$A = diag(\ldots 0, \overset{\overset{0}{\vee}}{0}, 1, 2, 2, \ldots), \qquad B = diag(\ldots 0, \overset{\overset{0}{\vee}}{1}, 1, 2, 2, \ldots),$$

and let U be the bilateral shift on $\ell^2(\mathbf{Z})$.

In this case, we have:

$$U^*AU = diag(\ldots 0, 0, \overset{\overset{0}{\vee}}{1}, 2, 2, \ldots).$$

We hence get $A \preccurlyeq B \preccurlyeq U^*AU$. This means $A \preccurlyeq B \overset{u}{\preccurlyeq} A$. But since the dimensions of eigenspaces of $A - 1$ and $B - 1$ are different, A and B are not unitarily equivalent. This means $A \preccurlyeq U^*AU$ but $A \neq U^*AU$; this is a counterexample for Theorem 2.13 of [8]. Incidentally, $P := diag(\ldots 0, \overset{\overset{0}{\vee}}{1}, 1, \ldots)$ is a projection and $P \preccurlyeq U^*PU$ but $P \neq U^*PU$. This is a counterexample for Lemma 2.10 of [8] as well.

We next consider (1). In Theorem 2.8 of [8] it was tried to extend this; but the proof was as follows:

For $0 < A \leqslant B$ and for any $r > 0$ there is a unitary V such that $f(g(A)^r) \leqslant V^* f(g(B)^r)V$. $0 \leqslant A \leqslant B$ ensures $0 < A + \epsilon \leqslant B + \epsilon$ for all $\epsilon > 0$, so for any $r > 0$, there is a unitary V such that $f(g(A+\epsilon)^r) \leqslant V^* f(g(B+\epsilon)^r)V$. By letting $\epsilon \to 0$, $f(g(A)^r) \leqslant V^* f(g(B)^r)V$; and hence $f(g(A)^r) \overset{u}{\leqslant} f(g(B)^r)$ for any operator convex function f and any operator monotone function g.

But V depends on $\epsilon > 0$, so authors should have referred to the limit of V; it is not obvious whether or not a sequence of unitary operators converges to a non-trivial unitary operator. In this sense, the proof is not complete.

From now on, we take a different approach to this problem. We start with a characterization of $A \overset{u}{\leqslant} B$ for $A, B \geqslant 0$.

Lemma 1. *Let $A, B \geqslant 0$. Then $A \overset{u}{\leqslant} B$ if and only if there is a contraction X such that $A = X^*BX$ and $\dim \mathbf{N}(B^{1/2}X) = \dim \mathbf{N}(X^*B^{1/2}) \leqslant \infty$, where $\mathbf{N}(X) = \{\mathbf{x}\colon X\mathbf{x} = \mathbf{0}\}$.*

Proof. Assume there is a unitary U such that $A \leqslant U^*BU$. Then there is a contraction Y satisfying $A^{1/2} = YU^*B^{1/2}U$. This implies $A = YU^*BUY^*$, because $A^{1/2}$ is self-adjoint. Putting $X = UY^*$, we clearly get $A = X^*BX$, $\|X\| \leqslant 1$, and:

$$\dim \mathbf{N}\big(X^*B^{1/2}\big) = \dim \mathbf{N}\big(YU^*B^{1/2}\big) = \dim \mathbf{N}\big(A^{1/2}\big) = \dim \mathbf{N}\big(B^{1/2}X\big).$$

We next show the converse implication. Let $B^{1/2}X = V|B^{1/2}X|$ be the polar decomposition. By the assumption, V has a unitary extension U. Since $A^{1/2} = |B^{1/2}X|$, in virtue of $V|B^{1/2}X| = U|B^{1/2}X|$, we have:

$$UAU^* = U\big|B^{1/2}X\big|\big|B^{1/2}X\big|U^* = B^{1/2}XX^*B^{1/2} \leqslant B,$$

which yields $A \leqslant U^*BU$. This is the required result. □

We remark that if B is invertible, then the condition on the null spaces in Lemma 1 reduces to $\dim \mathbf{N}(X) = \dim \mathbf{N}(X^*)$, which always holds if X is self-adjoint or invertible. The geometric mean:

$$B^{-1}\#A := B^{-1/2}\big(B^{1/2}AB^{1/2}\big)^{1/2}B^{-1/2}$$

satisfies $A = (B^{-1}\#A)B(B^{-1}\#A)$. We therefore get:

Example 2. Suppose $A, B \geqslant 0$ and B is invertible. Then:

$$B^{-1}\#A \leqslant 1 \quad \Longrightarrow \quad A \overset{u}{\leqslant} B.$$

A continuous function $g(t)$ defined on an interval I is called an *operator convex* function if $g(sA + (1-s)B) \leqslant sg(A) + (1-s)g(B)$ for $0 < s < 1$ and for A and B with spectra in I. A continuous function $f(t)$ defined on I is called an *operator monotone* function if $f(A) \leqslant f(B)$ whenever $A \leqslant B$. A power function t^a is operator monotone on $[0,\infty)$ if $0 < a \leqslant 1$, and operator convex there if $1 \leqslant a \leqslant 2$. It is clear that $A \overset{u}{\leqslant} B$ implies $f(A) \overset{u}{\leqslant} f(B)$ for an operator monotone function $f(t)$.

Proposition 1. *Let $g_i(t) \geqslant 0$ $(i = 1, 2, \ldots, n)$ be an operator monotone function or an operator convex function on $[0,\infty)$ with $g_i(0) = 0$. Let $A, B \geqslant 0$ and B be invertible. Then $0 \leqslant A \overset{u}{\leqslant} B$ implies:*

$$(g_n \circ \cdots \circ g_1)(A) \overset{u}{\leqslant} (g_n \circ \cdots \circ g_1)(B),$$

where $\circ$ is the symbol of composition. In particular, $A^a \overset{u}{\leqslant} B^a$ for every $a > 0$.

Proof. We first show $g_1(A) \overset{u}{\leqslant} g_1(B)$. This is clear if $g_1(t)$ is operator monotone, so we assume that $g_1(t)$ is operator convex. We may also assume $g_1(t)$ is not a constant. By Lemma 1 there is a contraction X such that $A = X^*BX$ and $\dim \mathbf{N}(X) = \dim \mathbf{N}(X^*)$. By the Hansen–Pedersen inequality [5]:

$$g_1(A) = g_1\big(X^*BX\big) \leqslant X^*g_1(B)X.$$

Since $0 \leqslant g_1(t)$ is strictly increasing, $g_1(B)$ is invertible as well. By Lemma 1:

$$X^*g_1(B)X \overset{u}{\leqslant} g_1(B).$$

We therefore get $g_1(A) \overset{u}{\leqslant} g_1(B)$. By induction, we get the required result. Substituting power functions for $g_i(t)$ yields the last inequality. □

We remark that the last statement does not imply that there is a common unitary U such that $A^a \leqq U^*B^aU$ for every $a > 0$, namely $A \overset{u}{\preccurlyeq} B$. To extend this proposition, we use a few symbols induced in [16]. Let I be a right open interval, namely $I = (a,b)$ or $I = [a,b)$, and $\mathbb{P}_+^{-1}(I)$ the set of all increasing and continuous functions $h(t)$ on I such that $\lim_{t\to a+0} h(t) = 0$, $\lim_{t\to b-0} h(t) = \infty$ and h^{-1} is operator monotone. Let $\mathbb{LP}_+(I)$ denote the set of all functions $h(t)$ such that $h(t) > 0$ on the interior of I and $\log h(t)$ is operator monotone there. e^t obviously belongs to $\mathbb{P}_+^{-1}(-\infty,\infty) \cap \mathbb{LP}_+(-\infty,\infty)$. In Theorem 2.11 of [16] (also see [14,15]), it was shown that $\mathbb{LP}_+(I)\cdot\mathbb{P}_+^{-1}(I) \subseteq \mathbb{P}_+^{-1}(I)$; hence for $a \geqslant 1$, $b > 0$: $t^a e^{bt} \in \mathbb{P}_+^{-1}[0,\infty) \cap \mathbb{LP}_+[0,\infty)$. If $g(t)$ is an operator convex function on $[0,\infty)$ with $g(0) = 0$, then: $g(t) \in \mathbb{P}_+^{-1}[0,\infty) \cap \mathbb{LP}_+[0,\infty)$ since $g(t)/t$ is operator monotone. Moreover, in Corollary 4.3 of [16] it was proved that:

if $h \in \mathbb{P}_+^{-1}(I) \cap \mathbb{LP}_+(I)$, $p \geqslant 1$, $r > 0$ and $0 < \alpha \leqslant \frac{r}{p+r}$ then for A, B with $\sigma(A), \sigma(B) \subset I$

$$A \leqslant B \quad \Rightarrow \quad \begin{cases} \big(h(A)^{\frac{r}{2}}h(A)^p h(A)^{\frac{r}{2}}\big)^\alpha \leqslant \big(h(A)^{\frac{r}{2}}h(B)^p h(A)^{\frac{r}{2}}\big)^\alpha, \\ \big(h(B)^{\frac{r}{2}}h(A)^p h(B)^{\frac{r}{2}}\big)^\alpha \leqslant \big(h(B)^{\frac{r}{2}}h(B)^p h(B)^{\frac{r}{2}}\big)^\alpha. \end{cases} \tag{2}$$

We are now in the position to state the main result.

Theorem 1. *Let $h \in \mathbb{P}_+^{-1}(I) \cap \mathbb{LP}_+(I)$ and $\sigma(A), \sigma(B) \subset I$. Suppose $h(B)$ is invertible. Then, for every $a > 0$:*

$$A \overset{u}{\leqslant} B \quad \Longrightarrow \quad h(A)^a \overset{u}{\leqslant} h(B)^a. \tag{3}$$

In particular, if $0 \leqslant A \overset{u}{\leqslant} B$ and B is invertible, then for $a > 0$, $b \geqslant 0$:

$$A^a e^{bA} \overset{u}{\leqslant} B^a e^{bB}; \tag{4}$$

moreover, for $a_i > 0$, $b_i \geqslant 0$, $c_i \geqslant 0$:

$$\sum_{i=1}^n c_i A^{a_i} e^{b_i A} \overset{u}{\leqslant} \sum_{i=1}^n c_i B^{a_i} e^{b_i B}. \tag{5}$$

Proof. We first show (3) in the case $a=1$. Suppose $U^*AU \leqslant B$. Put $p=r=1$ and $\alpha=1/2$ in the second inequality of (2) to get: $(h(B)^{\frac{1}{2}}h(U^*AU)h(B)^{\frac{1}{2}})^{\frac{1}{2}} \leqslant h(B)$, which means $h(B)^{-1}\#h(U^*AU) \leqslant 1$. By Example 2 we have $h(U^*AU) \overset{u}{\leqslant} h(B)$, which implies (3). In the case of $a>1$, $h(t)^a$ itself belongs to $\mathbb{P}_+^{-1}(I) \cap \mathbb{LP}_+(I)$. In the case of $0<a<1$, $h(A)^a \overset{u}{\leqslant} h(B)^a$ follows from $h(A) \overset{u}{\leqslant} h(B)$, because t^a is operator monotone. We have consequently shown (3). Since $te^{\frac{b}{a}t} \in \mathbb{P}_+^{-1}[0,\infty) \cap \mathbb{LP}_+[0,\infty)$, (3) deduces (4). To show (5) put $a=\max\{1,a_1,\dots,a_n\}$ and $b=\max\{b_i\colon\ 1\leqslant i\leqslant n\}$. Then by (4) there is a unitary U such that $A^ae^{bA} \leqslant U^*B^ae^{bB}U$. Since for each i there is an operator monotone function ϕ_i such that $t^{a_i}e^{b_it}=\phi_i(t^ae^{bt})$ (see Theorem 2.11 of [16]) we obtain:

$$A^{a_i}e^{b_iA}=\phi_i\big(A^ae^{bA}\big) \leqslant \phi_i\big(U^*B^ae^{bB}U\big)=U^*B^{a_i}e^{b_iB}U$$

for each i. This yields (5) since $c_i \geqslant 0$. □

We remark that (3) involves (1) and Proposition 1, because $e^t \in \mathbb{P}_+^{-1}(-\infty,\infty) \cap \mathbb{LP}_+(-\infty,\infty)$ and $g_i(t) \in \mathbb{P}_+^{-1}[0,\infty) \cap \mathbb{LP}_+[0,\infty)$.

Let ϕ be a unital positive linear map on $B_h(\mathfrak{H})$. Then, Choi [4] has shown that for $A \geqslant 0$ and an operator convex function $g(t) \geqslant 0$ on $(0,\infty)$:

$$g\big(\phi(A)\big) \leqslant \phi\big(g(A)\big). \tag{6}$$

In particular, $\phi(A)^2 \leqslant \phi(A^2)$ [6]. Of course, $\phi(A)^a \leqslant \phi(A^a)$ does not necessarily hold for $a>2$. However we have the following:

Proposition 2. *Let ϕ be a unital positive linear map, and let $g_i(t) \geqslant 0$ be operator convex functions on $[0,\infty)$ with $g_i(0)=0$. Then for invertible $A \geqslant 0$:*

$$(g_n \circ \dots \circ g_1)\big(\phi(A)\big) \overset{u}{\leqslant} \phi\big((g_n \circ \dots \circ g_1)(A)\big).$$

In particular, for $a>2$:

$$\big(\phi(A)\big)^a \overset{u}{\leqslant} \phi\big(A^a\big).$$

Proof. By (6) we first get $g_1(\phi(A)) \leqslant \phi(g_1(A))$. Since $\phi(g_1(A))$ is invertible, it follows from Proposition 1 and (6) that:

$$g_2\big(g_1\big(\phi(A)\big)\big) \overset{u}{\leqslant} g_2\big(\phi\big(g_1(A)\big)\big) \leqslant \phi\big(g_2\big(g_1(A)\big)\big).$$

This implies that $g_2(g_1(\phi(A))) \overset{u}{\leqslant} \phi(g_2(g_1(A)))$. By induction, we obtain the desired result. □

We remark that Bourin and Lee [3] have shown that if g is a monotone convex function on $\mathbf{R}$ and A is a bounded self-adjoint operator, then for an arbitrary $0<r\in\mathbf{R}$:

$$g\big(\phi(A)\big) \overset{u}{\leqslant} \phi\big(g(A)\big)+rI.$$

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