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Expression of Dirichlet boundary conditions in terms of the Cauchy–Green tensor field

Expression de conditions aux limites de Dirichlet en fonction du champ de tenseurs de Cauchy–Green

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ABSTRACT

In a previous work, it was shown how the Cauchy–Green tensor field $\boldsymbol{C} := \nabla\boldsymbol{\Phi}^T\nabla\boldsymbol{\Phi} \in W^{2,s}(\Omega;\mathbb{S}^3_>)$, $s > 3/2$, can be considered as the sole unknown in the homogeneous Dirichlet problem of nonlinear elasticity posed over a domain $\Omega \subset \mathbb{R}^3$, instead of the deformation $\boldsymbol{\Phi} \in W^{3,s}(\Omega;\mathbb{R}^3)$ in the usual approach. The purpose of this Note is to show that the same approach applies as well to the Dirichlet–Neumann problem. To this end, we show how the boundary condition $\boldsymbol{\Phi} = \boldsymbol{\Phi}_0$ on a portion Γ_0 of the boundary of Ω can be recast, again as boundary conditions on Γ_0, but this time expressed only in terms of the new unknown $\boldsymbol{C} \in W^{2,s}(\Omega;\mathbb{S}^3_>)$.

RÉSUMÉ

Dans un travail antérieur, on a montré comment le champ $\boldsymbol{C} := \nabla\boldsymbol{\Phi}^T\nabla\boldsymbol{\Phi} \in W^{2,s}(\Omega;\mathbb{S}^3_>)$, $s > 3/2$, des tenseurs de Cauchy–Green peut être considéré comme la seule inconnue dans le problème de Dirichlet homogène pour l'élasticité non linéaire posé sur un domaine $\Omega \subset \mathbb{R}^3$, au lieu de la déformation $\boldsymbol{\Phi} \in W^{3,s}(\Omega;\mathbb{R}^3)$ dans l'approche habituelle. L'objet de cette Note est de montrer que la même approche s'applique aussi bien au problème de Dirichlet–Neumann. À cette fin, nous montrons comment la condition aux limites $\boldsymbol{\Phi} = \boldsymbol{\Phi}_0$ sur une portion Γ_0 de la frontière de Ω peut être ré-écrite, à nouveau sous forme de conditions aux limites sur Γ_0, mais exprimées cette fois uniquement en fonction de la nouvelle inconnue $\boldsymbol{C} \in W^{2,s}(\Omega;\mathbb{S}^3_>)$.

1. Preliminaries

Greek indices, resp. Latin indices, range over the set $\{1,2\}$, resp. $\{1,2,3\}$. The summation convention with respect to repeated indices is used in conjunction with these rules.

The notations $\mathbb{S}^3$, $\mathbb{S}^3_>$, and $\mathbb{O}^3_+$, respectively designate the space of all symmetric matrices, the set of all positive-definite symmetric matrices, and the set of all proper orthogonal matrices, of order 3. The notation $f|_A$ designates the restriction

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to the set A of a function f defined over a set that contains A. Given a normed vector space X, the notation $\mathcal{L}^2_{\mathrm{sym}}(X \times X)$ designates the space of all continuous symmetric bilinear forms defined on the product $X \times X$.

The Euclidean norm, the exterior product, the dyadic product, and the inner product of vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ are respectively denoted $|\mathbf{u}|$, $\mathbf{u} \wedge \mathbf{v}$, $\mathbf{u} \otimes \mathbf{v}$, and $\mathbf{u} \cdot \mathbf{v}$. The inner product of two $m \times m$ tensors $\boldsymbol{e}$ and $\boldsymbol{\tau}$ is denoted and defined by $\boldsymbol{e} : \boldsymbol{\tau} = \mathrm{tr}(\boldsymbol{e}^T \boldsymbol{\tau})$.

The set of k-times continuously differentiable functions from an open subset $U \subset X$ of a normed vector space X into a subset $V \subset Y$ of a normed vector space Y is denoted $\mathcal{C}^k(U; V)$. The set $\mathcal{C}^k(\overline{U}; V)$ is defined as the subset of the space $\mathcal{C}^k(U; V)$ that consists of all functions $f \in \mathcal{C}^k(U; V)$ that, together with all their partial derivatives of order $\leqslant k$, possess continuous extensions to the closure $\overline{U}$ of U, the extension of f being in addition with values in V.

Throughout this Note, the notation Ω designates a *bounded* and *connected* open subset of $\mathbb{R}^3$, whose boundary $\Gamma := \partial\Omega$ is of class $\mathcal{C}^4$. This means that there exists a finite number N of open sets $\omega^k \subset \mathbb{R}^2$ and of injective immersions $\boldsymbol{\theta}^k \in \mathcal{C}^4(\omega^k; \mathbb{R}^3)$, $k = 1, 2, \ldots, N$, such that $\Gamma = \bigcup_{k=1}^N \boldsymbol{\theta}^k(\omega^k)$. It also implies that there exists $\varepsilon > 0$ such that the mappings $\boldsymbol{\Theta}^k \in \mathcal{C}^3(U^k; \mathbb{R}^3)$ defined by:

$$\boldsymbol{\Theta}^k(y, y_3) := \boldsymbol{\theta}^k(y) + y_3 \boldsymbol{a}_3^k(y) \quad \text{for all } (y, y_3) \in U^k := \omega^k \times (-\varepsilon, \varepsilon),$$

where $\boldsymbol{a}_3^k$ denotes the unit inner normal vector field along the portion $\boldsymbol{\theta}^k(\omega^k)$ of the boundary of Ω, are $\mathcal{C}^3$-diffeomorphisms onto their image (cf. [3, Theorem 4.1-1]). Thus the mappings $\{\boldsymbol{\Theta}^k;\ 1 \leqslant k \leqslant N\}$ form an atlas of local charts for the set $\Omega_\varepsilon := \{x \in \overline{\Omega};\ \mathrm{dist}(x, \Gamma) < \varepsilon\} \subset \overline{\Omega}$, while the mappings $\{\boldsymbol{\theta}^k;\ 1 \leqslant k \leqslant N\}$ form an atlas of local charts for the surface $\Gamma = \partial\Omega \subset \mathbb{R}^3$. When no confusion should arise, we will drop the explicit dependence on k for notational brevity.

Generic points in ω, in $U = \omega \times (-\varepsilon, \varepsilon)$, and in $\overline{\Omega}$, are respectively denoted $y = (y_\alpha)$, (y, y_3), and $x = (x_i)$. Partial derivatives with respect to y_i are denoted $\partial_i := \partial/\partial y_i$, while partial derivatives with respect to x_i are denoted $\partial/\partial x_i$. The gradient of a vector field $\boldsymbol{\Phi} = (\Phi_i) : \overline{\Omega} \to \mathbb{R}^3$ is the 3×3 matrix field denoted and defined by $\nabla\boldsymbol{\Phi} := (\partial\Phi_i/\partial x_j)$, with i as its row index.

The tangent plane $T_x\Gamma$ of the surface $\Gamma \subset \mathbb{R}^3$ at the point $x \in \Gamma$ will be identified with the subspace of $\mathbb{R}^3$ spanned by the vectors $\boldsymbol{a}_\alpha(y) := \partial_\alpha\boldsymbol{\theta}(y)$, where $y = \boldsymbol{\theta}^{-1}(y)$. The vectors $\boldsymbol{a}^\beta(y)$ of the dual basis of the tangent plane are those defined by the relations $\boldsymbol{a}^\beta(y) \cdot \boldsymbol{a}_\alpha(y) = \delta_\alpha^\beta$. The unit inner normal vector to $T_x\Gamma$ is defined by $\boldsymbol{a}_3(y) = \boldsymbol{a}^3(y) := (\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y))/|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)|$ (to ensure that the vector $\boldsymbol{a}_3(y)$ points toward the interior of Ω, it suffices to exchange if necessary the coordinates y_1 and y_2).

The unit outer normal vector field along the boundary Γ of Ω is denoted $\boldsymbol{n}$; thus $\boldsymbol{n}(x) = -\boldsymbol{a}_3(y)$, $y = \boldsymbol{\theta}^{-1}(x)$, in a local chart.

The tangent space $T_x\mathbb{R}^3$ of the Euclidean space $\mathbb{R}^3$ at the point $x \in \mathbb{R}^3$ will be identified with $\mathbb{R}^3$ by means of the basis formed by the vectors $\boldsymbol{g}_i(y, y_3) := \partial_i\boldsymbol{\Theta}(y, y_3)$, where $(y, y_3) = \boldsymbol{\Theta}^{-1}(x)$. The vectors of the dual basis are those defined by the relations $\boldsymbol{g}^j(y, y_3) \cdot \boldsymbol{g}_i(y, y_3) = \delta_i^j$. Note that:

$$\boldsymbol{g}_\alpha(y, y_3) = \boldsymbol{a}_\alpha(y) + y_3 \partial_\alpha \boldsymbol{a}_3(y) \quad \text{and} \quad \boldsymbol{g}_3(y, y_3) = \boldsymbol{a}_3(y).$$

Let $\Gamma_0 \subset \Gamma$ denote a *relatively open subset of* Γ. Since Γ is a manifold of class $\mathcal{C}^4$, so is Γ_0. It follows that functions, vector fields, and tensor fields, of class $\mathcal{C}^k$, $0 \leqslant k \leqslant 4$, can be defined on Γ_0. The Lebesgue and Sobolev spaces on Γ_0 used in this paper are defined as in, e.g., Aubin [1].

Spaces of vector fields, resp. symmetric tensor fields, with values in $\mathbb{R}^3$, resp. in $\mathbb{S}^3$, are defined by using a given Cartesian basis $\{\hat{\boldsymbol{e}}^i,\ 1 \leqslant i \leqslant 3\}$ in $\mathbb{R}^3$, resp. the basis $\{\frac{1}{2}(\hat{\boldsymbol{e}}^i \otimes \hat{\boldsymbol{e}}^j + \hat{\boldsymbol{e}}^j \otimes \hat{\boldsymbol{e}}^i), 1 \leqslant i, j \leqslant 3\}$ in $\mathbb{S}^3$. They will be denoted by bold letters and by capital Roman letters, respectively.

Given an orientation-preserving immersion $\boldsymbol{\Phi} : \overline{\Omega} \to \mathbb{R}^3$ of class $\mathcal{C}^1$, that is, a mapping $\boldsymbol{\Phi} \in \boldsymbol{C}^1(\overline{\Omega}) := \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^3)$ such that $\det(\nabla\boldsymbol{\Phi}(x)) > 0$ for all $x \in \overline{\Omega}$, the *Cauchy–Green*, or *metric, tensor field* induced by $\boldsymbol{\Phi}$ is the field $\boldsymbol{C} := \nabla\boldsymbol{\Phi}^T \nabla\boldsymbol{\Phi} \in \mathcal{C}^0(\overline{\Omega}; \mathbb{S}^3_>)$. Each Cauchy–Green tensor field $\boldsymbol{C} \in \mathcal{C}^0(\overline{\Omega}; \mathbb{S}^3_>)$ defines a Riemannian metric on $\overline{\Omega}$ by means of the bilinear forms:

$$\big(\boldsymbol{C}(x)\big)(\boldsymbol{u}, \boldsymbol{v}) := \boldsymbol{u}^T \boldsymbol{C}(x) \boldsymbol{v} \quad \text{for all } (\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{R}^3 \times \mathbb{R}^3,\ x \in \overline{\Omega}.$$

Complete proofs and complements will be found in [6].

2. Fundamental forms of the surface Γ_0

Given an orientation-preserving immersion $\boldsymbol{\Phi} \in \boldsymbol{C}^2(\overline{\Omega})$, the restriction $\boldsymbol{\varphi} := \boldsymbol{\Phi}|_{\Gamma_0} \in \boldsymbol{C}^2(\overline{\Gamma}_0)$ is an immersion of $\overline{\Gamma}_0$ into $\mathbb{R}^3$. The *first and second fundamental forms* induced by $\boldsymbol{\varphi}$ are then respectively defined in each local chart by:

$$\begin{aligned}
&\mathbf{a}(\boldsymbol{\varphi}) \circ \boldsymbol{\theta} = a_{\alpha\beta}(\boldsymbol{\varphi})\, \boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta, \quad \text{where } a_{\alpha\beta}(\boldsymbol{\varphi}) := \boldsymbol{a}_\alpha(\boldsymbol{\varphi}) \cdot \boldsymbol{a}_\beta(\boldsymbol{\varphi}),\\
&\mathbf{b}(\boldsymbol{\varphi}) \circ \boldsymbol{\theta} = b_{\alpha\beta}(\boldsymbol{\varphi})\, \boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta, \quad \text{where } b_{\alpha\beta}(\boldsymbol{\varphi}) := \partial_\alpha \boldsymbol{a}_\beta(\boldsymbol{\varphi}) \cdot \boldsymbol{a}_3(\boldsymbol{\varphi}) = -\boldsymbol{a}_\alpha(\boldsymbol{\varphi}) \cdot \partial_\beta \boldsymbol{a}_3(\boldsymbol{\varphi}),
\end{aligned} \tag{1}$$

the vector fields $\boldsymbol{a}_i(\boldsymbol{\varphi})$ and $\boldsymbol{a}^j(\boldsymbol{\varphi})$ being defined by:

$$\boldsymbol{a}_\alpha(\boldsymbol{\varphi}) := \partial_\alpha(\boldsymbol{\varphi} \circ \boldsymbol{\theta}), \qquad \boldsymbol{a}_3(\boldsymbol{\varphi}) := \frac{\boldsymbol{a}_1(\boldsymbol{\varphi}) \wedge \boldsymbol{a}_2(\boldsymbol{\varphi})}{|\boldsymbol{a}_1(\boldsymbol{\varphi}) \wedge \boldsymbol{a}_2(\boldsymbol{\varphi})|}, \quad \text{and} \quad \boldsymbol{a}_i(\boldsymbol{\varphi}) \cdot \boldsymbol{a}^j(\boldsymbol{\varphi}) = \delta_i^j.$$

Given any metric tensor field $\boldsymbol{C} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{S}^3_>)$, let:

$$\boldsymbol{N}_\sharp(\boldsymbol{C}) := \frac{\boldsymbol{C}^{-1}\boldsymbol{N}}{[\boldsymbol{N}^T\boldsymbol{C}^{-1}\boldsymbol{N}]^{1/2}} \in \boldsymbol{C}^1(\overline{\Omega}) \quad \text{and} \quad \boldsymbol{n}_\sharp(\boldsymbol{C}) := \boldsymbol{N}_\sharp|_{\Gamma_0} \in \boldsymbol{C}^1(\overline{\Gamma}_0),$$

where $\boldsymbol{N} \in \boldsymbol{C}^3(\overline{\Omega})$ denotes any $\mathcal{C}^3$-extension to $\overline{\Omega}$ of the unit outer normal vector field $\boldsymbol{n} \in \boldsymbol{C}^3(\Gamma)$ to the boundary of Ω defined in Section 1. Then $\boldsymbol{n}_\sharp := \boldsymbol{n}_\sharp(\boldsymbol{C})$ is a unit outer normal vector field to Γ_0 *with respect to the metric* $\boldsymbol{C}$, that is:

$$\boldsymbol{n}_\sharp(x)^T\boldsymbol{C}(x)\mathbf{t} = 0 \quad \text{for all } \mathbf{t} \in T_x\Gamma_0, \qquad \boldsymbol{n}_\sharp(x)^T\boldsymbol{C}(x)\boldsymbol{n}_\sharp(x) = 1, \quad \text{and} \quad \boldsymbol{n}_\sharp(x)^T\boldsymbol{C}(x)\boldsymbol{n}(x) > 0.$$

Note that the fields $\boldsymbol{N}_\sharp(\boldsymbol{C})$ and $\boldsymbol{n}_\sharp(\boldsymbol{C})$ will be simply abbreviated to $\boldsymbol{N}_\sharp$ and $\boldsymbol{n}_\sharp$, respectively, when no confusion should arise. The *first and second fundamental forms of the surface* Γ_0 induced by $\boldsymbol{C}$ are then the tensor fields:

$$\mathbf{a}^\sharp(\boldsymbol{C}) : x \in \Gamma_0 \to \big(\mathbf{a}^\sharp(\boldsymbol{C})\big)(x) \in \mathcal{L}^2_{\mathrm{sym}}(T_x\Gamma_0 \times T_x\Gamma_0),$$
$$\mathbf{b}^\sharp(\boldsymbol{C}) : x \in \Gamma_0 \to \big(\mathbf{b}^\sharp(\boldsymbol{C})\big)(x) \in \mathcal{L}^2_{\mathrm{sym}}(T_x\Gamma_0 \times T_x\Gamma_0),$$

defined in a local chart by:

$$\mathbf{a}^\sharp(\boldsymbol{C}) \circ \boldsymbol{\theta} = a^\sharp_{\alpha\beta}(\boldsymbol{C})\,\boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta, \quad \text{where } a^\sharp_{\alpha\beta}(\boldsymbol{C}) := C_{\alpha\beta}|_{\omega\times\{0\}},$$
$$\mathbf{b}^\sharp(\boldsymbol{C}) \circ \boldsymbol{\theta} = b^\sharp_{\alpha\beta}(\boldsymbol{C})\,\boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta, \quad \text{where } b^\sharp_{\alpha\beta}(\boldsymbol{C}) := \frac{1}{2}\big(C_{\alpha i}\partial_\beta N^i_\sharp + C_{\beta i}\partial_\alpha N^i_\sharp + N^i_\sharp\partial_i C_{\alpha\beta}\big)\big|_{\omega\times\{0\}}, \tag{2}$$

where the functions C_{ij} and $N^i_\sharp$ are defined by the relations:

$$\boldsymbol{C} \circ \boldsymbol{\Theta} = C_{ij}\,\boldsymbol{g}^i \otimes \boldsymbol{g}^j \quad \text{and} \quad \boldsymbol{N}_\sharp \circ \boldsymbol{\Theta} = N^i_\sharp \boldsymbol{g}_i \quad \text{in } \omega \times [0, \varepsilon).$$

Note that the definition of the second fundamental form $\mathbf{b}^\sharp(\boldsymbol{C})$ is independent of the choice (induced by the choice of an extension to $\overline{\Omega}$ of the unit outer normal vector field along the boundary of Ω) of the extension $\boldsymbol{N}_\sharp$ of $\boldsymbol{n}_\sharp$ to $\overline{\Omega}$.

Remark. In other words, the fundamental forms $\mathbf{a}^\sharp(\boldsymbol{C})$ and $\mathbf{b}^\sharp(\boldsymbol{C})$ are the restrictions of the *metric tensor* $\boldsymbol{C}$ and of its *Lie derivative* (cf., e.g., [2]) along the vector field $\boldsymbol{N}_\sharp$ to the subset $T_x\Gamma_0 \times T_x\Gamma_0$ of $T_x\mathbb{R}^3 \times T_x\mathbb{R}^3$, that is,

$$\big(\mathbf{a}^\sharp(\boldsymbol{C})\big)(x) = \boldsymbol{C}(x)|_{T_x\Gamma_0\times T_x\Gamma_0} \quad \text{and} \quad \big(\mathbf{b}^\sharp(\boldsymbol{C})\big)(x) = \frac{1}{2}(\mathcal{L}_{\boldsymbol{N}_\sharp}\boldsymbol{C})(x)|_{T_x\Gamma_0\times T_x\Gamma_0}, \quad x \in \Gamma_0. \qquad \square$$

The following theorem establishes the relation between the tensors fields defined by (1) and (2) when the vector field $\boldsymbol{\varphi}$ and the tensor field $\boldsymbol{C}$ are induced by the same orientation-preserving immersion $\boldsymbol{\Phi} : \overline{\Omega} \to \mathbb{R}^3$.

Theorem 1. *Given any orientation-preserving immersion* $\boldsymbol{\Phi} \in \boldsymbol{C}^2(\overline{\Omega})$*, let* $\boldsymbol{C} := \nabla\boldsymbol{\Phi}^T\nabla\boldsymbol{\Phi} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{S}^3_>)$ *and* $\boldsymbol{\varphi} := \boldsymbol{\Phi}|_{\Gamma_0} \in \boldsymbol{C}^2(\Gamma_0)$*. Then*

$$\mathbf{a}^\sharp(\boldsymbol{C}) = \mathbf{a}(\boldsymbol{\varphi}) \quad \text{in } \mathbb{C}^1(\Gamma_0) \quad \text{and} \quad \mathbf{b}^\sharp(\boldsymbol{C}) = \mathbf{b}(\boldsymbol{\varphi}) \quad \text{in } \mathbb{C}^0(\Gamma_0).$$

Sketch of proof. Proving the theorem amounts to proving the equalities:

$$a^\sharp_{\alpha\beta}(\boldsymbol{C}) = a_{\alpha\beta}(\boldsymbol{\varphi}) \quad \text{and} \quad b^\sharp_{\alpha\beta}(\boldsymbol{C}) = b_{\alpha\beta}(\boldsymbol{\varphi}) \quad \text{in } \omega,$$

in any local chart. The first equality follows from direct computations. The second equality follows from the observation that the relation $\boldsymbol{C} = \nabla\boldsymbol{\Phi}^T\nabla\boldsymbol{\Phi}$ implies that the vector field $\boldsymbol{g}^3 \in \boldsymbol{C}^1(\overline{\Omega})$ defined by the relations $\boldsymbol{g}^3(x) \cdot \partial_i(\boldsymbol{\Phi} \circ \boldsymbol{\Theta})(x)$ at each $x \in \overline{\Omega}$ satisfies $\boldsymbol{g}^3|_\Gamma = \boldsymbol{N}_\sharp|_\Gamma$. $\square$

3. An intrinsic formulation of the boundary conditions

As a consequence of Theorem 1, we now show how *a Dirichlet boundary condition imposed on the orientation-preserving immersion* $\boldsymbol{\Phi}$ *in the displacement-traction problem of nonlinear elasticity can be replaced by a boundary condition imposed on the Cauchy–Green tensor field* $\boldsymbol{C}$. The set of *proper isometries* of $\mathbb{R}^3$ appearing in the next theorem is defined by:

$$\mathbf{Rig}_+\big(\mathbb{R}^3\big) := \big\{\boldsymbol{\chi} : \mathbb{R}^3 \to \mathbb{R}^3;\ \text{there exist } \boldsymbol{c} \in \mathbb{R}^3 \text{ and } \boldsymbol{Q} \in \mathbb{O}^3_+ \text{ such that } \boldsymbol{\chi}(x) = \boldsymbol{c} + \boldsymbol{Q}x,\ x \in \mathbb{R}^3\big\}.$$

Theorem 2. *Let there be given an orientation-preserving immersion $\boldsymbol{\Phi}_0 \in \mathbf{C}^2(\overline{\Omega})$ and let $\boldsymbol{\varphi}_0 := \boldsymbol{\Phi}_0|_{\Gamma_0} \in \mathbf{C}^2(\Gamma_0)$. If an orientation-preserving immersion $\boldsymbol{\Phi} \in \mathbf{C}^2(\overline{\Omega})$ satisfies the boundary condition:*

$$\boldsymbol{\Phi}|_{\Gamma_0} = \boldsymbol{\Phi}_0|_{\Gamma_0} \quad \text{in } \mathbf{C}^2(\Gamma_0),$$

then the associated Cauchy–Green tensor field $\boldsymbol{C} = \nabla\boldsymbol{\Phi}^T\nabla\boldsymbol{\Phi} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{S}^3_>)$ satisfies the boundary conditions:

$$\mathbf{a}^\sharp(\boldsymbol{C}) = \mathbf{a}(\boldsymbol{\varphi}_0) \quad \text{in } \mathbb{C}^1(\Gamma_0) \quad \text{and} \quad \mathbf{b}^\sharp(\boldsymbol{C}) = \mathbf{b}(\boldsymbol{\varphi}_0) \quad \text{in } \mathbb{C}^0(\Gamma_0).$$

Assume in addition that Γ_0 is connected. If the Cauchy–Green tensor field $\boldsymbol{C} = \nabla\boldsymbol{\Phi}^T\nabla\boldsymbol{\Phi} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{S}^3_>)$ associated with an orientation-preserving immersion $\boldsymbol{\Phi} \in \mathbf{C}^2(\overline{\Omega})$ satisfies the boundary conditions:

$$\mathbf{a}^\sharp(\boldsymbol{C}) = \mathbf{a}(\boldsymbol{\varphi}_0) \quad \text{in } \mathbb{C}^1(\Gamma_0) \quad \text{and} \quad \mathbf{b}^\sharp(\boldsymbol{C}) = \mathbf{b}(\boldsymbol{\varphi}_0) \quad \text{in } \mathbb{C}^0(\Gamma_0), \tag{3}$$

then there exists a unique proper isometry $\boldsymbol{\chi} \in \mathbf{Rig}_+(\mathbb{R}^3)$ such that the immersion $(\boldsymbol{\chi} \circ \boldsymbol{\Phi}) \in \mathbf{C}^2(\overline{\Omega})$ satisfies the boundary condition:

$$(\boldsymbol{\chi} \circ \boldsymbol{\Phi})|_{\Gamma_0} = \boldsymbol{\Phi}_0|_{\Gamma_0} \quad \text{in } \mathbf{C}^2(\Gamma_0). \tag{4}$$

Sketch of proof. Let $\boldsymbol{C} = \nabla\boldsymbol{\Phi}^T\nabla\boldsymbol{\Phi} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{S}^3_>)$ and $\boldsymbol{\varphi} := \boldsymbol{\Phi}|_{\Gamma_0} \in \mathbf{C}^2(\Gamma_0)$ be defined by a same orientation-preserving immersion $\boldsymbol{\Phi} \in \mathbf{C}^2(\overline{\Omega})$. Then Theorem 1 shows that:

$$\mathbf{a}^\sharp(\boldsymbol{C}) = \mathbf{a}(\boldsymbol{\varphi}) \quad \text{in } \mathbb{C}^1(\Gamma_0) \quad \text{and} \quad \mathbf{b}^\sharp(\boldsymbol{C}) = \mathbf{b}(\boldsymbol{\varphi}) \quad \text{in } \mathbb{C}^0(\Gamma_0).$$

The conclusion follows by combining these relations with the classical rigidity lemma on a surface (see, e.g., [3] or [4]), which reads as follows: *If two immersions $\boldsymbol{\varphi} \in \mathbf{C}^2(\Gamma_0)$ and $\boldsymbol{\varphi}_0 \in \mathbf{C}^2(\Gamma_0)$ satisfy $\mathbf{a}(\boldsymbol{\varphi}) = \mathbf{a}(\boldsymbol{\varphi}_0)$ and $\mathbf{b}(\boldsymbol{\varphi}) = \mathbf{b}(\boldsymbol{\varphi}_0)$ on Γ_0, and if Γ_0 is connected, then there exists a proper isometry $\boldsymbol{\chi} \in \mathbf{Rig}_+(\mathbb{R}^3)$ such that $\boldsymbol{\varphi} = \boldsymbol{\chi} \circ \boldsymbol{\varphi}_0$.* □

Remark. The assumption that Γ_0 is *connected* is essential, as illustrated by the following counterexample: assume that $\Gamma_0 = \Gamma_{0,1} \cup \Gamma_{0,2}$ with $\overline{\Gamma}_{0,1} \cap \overline{\Gamma}_{0,2} = \emptyset$. Let $\boldsymbol{\Phi}_0 = \boldsymbol{id}$ and $\boldsymbol{C} = \nabla\boldsymbol{\Phi}^T\nabla\boldsymbol{\Phi} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{S}^3_>)$, where $\boldsymbol{\Phi} \in \mathbf{C}^2(\overline{\Omega})$ is an orientation-preserving immersion such that $\boldsymbol{\Phi} = \boldsymbol{\chi}_1$ in a neighborhood of $\Gamma_{0,1}$, and $\boldsymbol{\Phi} = \boldsymbol{\chi}_2$ in a neighborhood of $\Gamma_{0,2}$, where $\boldsymbol{\chi}_1 \neq \boldsymbol{\chi}_2$ are two proper isometries of $\mathbb{R}^3$. Then the boundary condition (3) is clearly satisfied, while (4) is not. □

4. Extension to Sobolev spaces

The results of Sections 2 and 3 can be extended to orientation-preserving immersions and Cauchy–Green tensor fields with components in Sobolev spaces with sufficient regularity, so as to ensure that the fundamental forms induced by the immersion $\boldsymbol{\varphi}$ and by the metric tensor field $\boldsymbol{C}$ are well defined and that the rigidity theorem on a surface (see the proof of Theorem 2) still holds.

In all that follows, the real numbers $s > 3/2$ and $p > 2$ are such that the trace operator from $W^{1,s}(\Omega)$ into $L^p(\Gamma_0)$ is well defined. Since in this case the space $W^{2,s}(\Omega)$ is also an algebra, the following implication holds:

$$\boldsymbol{\Phi} \in \boldsymbol{W}^{3,s}(\Omega) \quad \text{and} \quad \det\nabla\boldsymbol{\Phi} > 0 \quad \text{in } \overline{\Omega} \quad \Rightarrow \quad \boldsymbol{C} := \nabla\boldsymbol{\Phi}^T\nabla\boldsymbol{\Phi} \in W^{2,s}\big(\Omega, \mathbb{S}^3_>\big) \quad \text{and} \quad \boldsymbol{\varphi} := \boldsymbol{\Phi}|_{\Gamma_0} \in \boldsymbol{W}^{2,p}(\Gamma_0).$$

The definition of the tensor fields $\mathbf{a}(\boldsymbol{\varphi})$ and $\mathbf{b}(\boldsymbol{\varphi})$ can then be extended to fields $\boldsymbol{\varphi} \in \boldsymbol{W}^{2,p}(\Gamma_0)$, in which case:

$$\mathbf{a}(\boldsymbol{\varphi}) \in \mathbb{W}^{1,p}(\Gamma_0) \quad \text{and} \quad \mathbf{b}(\boldsymbol{\varphi}) \in \mathbb{L}^p(\Gamma_0).$$

To see this, note that $\boldsymbol{W}^{2,p}(\Gamma_0) \subset \mathbf{C}^1(\overline{\Gamma}_0)$ by the Sobolev embedding theorem; hence the vector field $\boldsymbol{a}_3(\boldsymbol{\varphi})$ appearing in the definition of $\mathbf{b}(\boldsymbol{\varphi})$ (see Section 2) is well defined and belongs to the space $\boldsymbol{W}^{1,p}(\Gamma_0)$.

The definitions of the tensor fields $\mathbf{a}^\sharp(\boldsymbol{C})$ and $\mathbf{b}^\sharp(\boldsymbol{C})$ can also be extended to matrix fields $\boldsymbol{C} \in W^{2,s}(\Omega, \mathbb{S}^3_>)$, in which case:

$$\mathbf{a}^\sharp(\boldsymbol{C}) \in \mathbb{W}^{1,p}(\Gamma_0) \quad \text{and} \quad \mathbf{b}^\sharp(\boldsymbol{C}) \in \mathbb{L}^p(\Gamma_0).$$

To see this, note that $W^{2,s}(\Omega, \mathbb{S}^3_>) \subset \mathcal{C}^0(\overline{\Omega}; \mathbb{S}^3_>)$ by the Sobolev embedding theorem; hence the vector field $\boldsymbol{N}_\sharp = \boldsymbol{N}_\sharp(\boldsymbol{C})$ appearing in the definition of $\mathbf{b}^\sharp(\boldsymbol{C})$ (see Section 2) is well defined and belongs to the space $\boldsymbol{W}^{2,s}(\Omega)$.

The above observations allow us to generalize Theorems 1 and 2 as follows.

Theorem 3. *Given any orientation-preserving immersion $\boldsymbol{\Phi} \in \boldsymbol{W}^{3,s}(\Omega)$, let $\boldsymbol{C} := \nabla\boldsymbol{\Phi}^T\nabla\boldsymbol{\Phi} \in W^{2,s}(\Omega; \mathbb{S}^3_>)$ and $\boldsymbol{\varphi} := \boldsymbol{\Phi}|_{\Gamma_0} \in \boldsymbol{W}^{2,p}(\Gamma_0)$. Then:*

$$\mathbf{a}^\sharp(\boldsymbol{C}) = \mathbf{a}(\boldsymbol{\varphi}) \quad \text{in } \mathbb{W}^{1,p}(\Gamma_0) \quad \text{and} \quad \mathbf{b}^\sharp(\boldsymbol{C}) = \mathbf{b}(\boldsymbol{\varphi}) \quad \text{in } \mathbb{L}^p(\Gamma_0).$$

Proof. The proof follows from Theorem 1 combined with the density of the space $\mathbf{C}^3(\overline{\Omega})$ in $\boldsymbol{W}^{3,s}(\Omega)$. □

Theorem 4. *Let there be given an orientation-preserving immersion $\boldsymbol{\Phi}_0 \in \boldsymbol{W}^{3,s}(\Omega)$ and let $\boldsymbol{\varphi}_0 := \boldsymbol{\Phi}_0|_{\Gamma_0} \in \boldsymbol{W}^{2,p}(\Gamma_0)$. If an orientation-preserving immersion $\boldsymbol{\Phi} \in \boldsymbol{W}^{3,s}(\Omega)$ satisfies the boundary condition*:

$$\boldsymbol{\Phi}|_{\Gamma_0} = \boldsymbol{\Phi}_0|_{\Gamma_0} \quad \text{in } \boldsymbol{W}^{2,p}(\Gamma_0),$$

then the associated Cauchy–Green tensor field $\boldsymbol{C} = \nabla\boldsymbol{\Phi}^T\nabla\boldsymbol{\Phi} \in W^{2,s}(\Omega;\mathbb{S}^3_>)$ satisfies the boundary conditions:

$$\mathbf{a}^\sharp(\boldsymbol{C}) = \mathbf{a}(\boldsymbol{\varphi}_0) \quad \text{in } \mathbb{W}^{1,p}(\Gamma_0) \quad \text{and} \quad \mathbf{b}^\sharp(\boldsymbol{C}) = \mathbf{b}(\boldsymbol{\varphi}_0) \quad \text{in } \mathbb{L}^p(\Gamma_0).$$

Assume in addition that Γ_0 is connected. If the Cauchy–Green tensor field $\boldsymbol{C} = \nabla\boldsymbol{\Phi}^T\nabla\boldsymbol{\Phi} \in W^{2,s}(\Omega;\mathbb{S}^3_>)$ associated with an orientation-preserving immersion $\boldsymbol{\Phi} \in \boldsymbol{W}^{3,s}(\Omega)$ satisfies the boundary conditions:

$$\mathbf{a}^\sharp(\boldsymbol{C}) = \mathbf{a}(\boldsymbol{\varphi}_0) \quad \text{in } \mathbb{W}^{1,p}(\Gamma_0) \quad \text{and} \quad \mathbf{b}^\sharp(\boldsymbol{C}) = \mathbf{b}(\boldsymbol{\varphi}_0) \quad \text{in } \mathbb{L}^p(\Gamma_0),$$

then there exists a unique proper isometry $\boldsymbol{\chi} \in \mathbf{Rig}_+(\mathbb{R}^3)$ such that the immersion $(\boldsymbol{\chi}\circ\boldsymbol{\Phi}) \in \boldsymbol{W}^{3,s}(\Omega)$ satisfies the boundary condition:

$$(\boldsymbol{\chi}\circ\boldsymbol{\Phi})|_{\Gamma_0} = \boldsymbol{\Phi}_0|_{\Gamma_0} \quad \text{in } \boldsymbol{W}^{2,p}(\Gamma_0).$$

Sketch of proof. Let $\boldsymbol{C} = \nabla\boldsymbol{\Phi}^T\nabla\boldsymbol{\Phi} \in W^{2,s}(\Omega;\mathbb{S}^3_>)$ and $\boldsymbol{\varphi} := \boldsymbol{\Phi}|_{\Gamma_0} \in \boldsymbol{W}^{2,p}(\Gamma_0)$ be defined by a same orientation-preserving immersion $\boldsymbol{\Phi} \in \boldsymbol{W}^{3,s}(\Omega)$. Then Theorem 3 shows that:

$$\mathbf{a}^\sharp(\boldsymbol{C}) = \mathbf{a}(\boldsymbol{\varphi}) \quad \text{in } \mathbb{W}^{1,p}(\Gamma_0) \quad \text{and} \quad \mathbf{b}^\sharp(\boldsymbol{C}) = \mathbf{b}(\boldsymbol{\varphi}) \quad \text{in } \mathbb{L}^p(\Gamma_0).$$

The conclusion follows by combining these relations with the following version of the rigidity lemma on a surface, due to [7]: *If two immersions $\boldsymbol{\varphi} \in \boldsymbol{W}^{2,p}(\Gamma_0)$ and $\boldsymbol{\varphi}_0 \in \boldsymbol{W}^{2,p}(\Gamma_0)$ satisfy $\mathbf{a}(\boldsymbol{\varphi}) = \mathbf{a}(\boldsymbol{\varphi}_0)$ in $\mathbb{W}^{1,p}(\Omega)$ and $\mathbf{b}(\boldsymbol{\varphi}) = \mathbf{b}(\boldsymbol{\varphi}_0)$ in $\mathbb{L}^p(\Gamma_0)$, and if Γ_0 is connected, then there exists a proper isometry $\boldsymbol{\chi} \in \mathbf{Rig}_+(\mathbb{R}^3)$ such that $\boldsymbol{\varphi} = \boldsymbol{\chi}\circ\boldsymbol{\varphi}_0$ in $\boldsymbol{W}^{2,p}(\Gamma_0)$.* □

Remark. In the above, we have only considered Sobolev spaces $\boldsymbol{W}^{k,s}(\Omega)$ for some integer k. However, all results of this section hold as well for orientation-preserving immersions $\boldsymbol{\Phi} \in \boldsymbol{W}^{2+\frac{1}{p},p}(\Omega)$ for any $p > 2$, since then:

$$\boldsymbol{C} := \nabla\boldsymbol{\Phi}^T\nabla\boldsymbol{\Phi} \in W^{1+\frac{1}{p},p}\big(\Omega,\mathbb{S}^3_>\big) \quad \text{and} \quad \boldsymbol{\varphi} := \boldsymbol{\Phi}|_{\Gamma_0} \in \boldsymbol{W}^{2,p}(\Gamma_0),$$

which in turn implies that:

$$\mathbf{a}(\boldsymbol{\varphi}), \mathbf{a}^\sharp(\boldsymbol{C}) \in \mathbb{W}^{1,p}(\Gamma_0) \quad \text{and} \quad \mathbf{b}(\boldsymbol{\varphi}), \mathbf{b}^\sharp(\boldsymbol{C}) \in \mathbb{L}^p(\Gamma_0). \qquad \square$$

We refer to the extended article [6] for applications to *nonlinear elasticity* of the results presented in this Note. There, it will be shown in particular how the *Dirichlet–Neumann boundary value problem of three-dimensional nonlinear elasticity can be completely recast as a boundary value problem with the tensor field $\boldsymbol{C} = \nabla\boldsymbol{\Phi}^T\nabla\boldsymbol{\Phi}$ as the sole unknown*. Such a result thus complements the approach of [5], which was restricted to the *homogeneous pure Dirichlet problem* of three-dimensional nonlinear elasticity.

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