Algebraic Geometry

# Families of curves over $\mathbb{P}^{1}$ with 3 singular fibers * 

# Familles de courbes sur $\mathbb{P}^{1}$ avec trois fibres singulières 

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## A R T I C L E I N F O

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#### Abstract

Suppose $f: S \rightarrow \mathbb{P}^{1}$ is a fibration of genus $g$ with 3 singular fibers and two of them are semistable. We show that the Mordell-Weil group of $f$ is finite, the surface $S$ is rational and $2 g \leqslant-K_{S}^{2} \leqslant 4 g-4$. We construct some examples to show that such fibrations exist for infinitely many $g$.


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## R É S U M É

Soit $f: S \rightarrow \mathbb{P}^{1}$ une fibration de genre $g$ avec trois fibres singulières, dont deux d'entre elles sont semi-stables. Nous montrons que le groupe de Mordell-Weil de $f$ est fini, que la surface $S$ est rationnelle et que $2 g \leqslant-K_{S}^{2} \leqslant 4 g-4$. Nous construisons également des exemples montrant qu'il existe de telles fibrations pour une infinité de $g$.
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## 1. Introduction

Non-trivial semistable families $f: X \rightarrow \mathbb{P}^{1}$ of complex algebraic varieties over $\mathbb{P}^{1}$ with minimal number $s$ of singular fibers have some remarkable arithmetic and geometric properties. For example, suppose $f: S \rightarrow \mathbb{P}^{1}$ is a non-trivial family of semistable curves of genus $g \geqslant 1$ with $s$ singular fibers, Beauville proves that $s \geqslant 4$ (see [1]). If $g=1$ and $s=4$, then $f$ must be a modular family. There are exactly 6 such families (see [2]). If $g>1$, then $s \geqslant 5$ ([15], see also [5,9]), there are only several examples with $s=5, g=2$ or 3 . It is conjectured that the number of singular fibers is at least 6 if $g$ is big enough.

Denote by $s_{0}$ the number of those singular fibers with non-compact Jacobians. Viehweg and Zuo prove that $s_{0} \geqslant 4$, and $s_{0}=4$ implies that $f$ is a Shimura family defined over an algebraic number field (see [19]). Kukulies [4] shows that $s_{0} \geqslant 5$ if $g$ is big enough. According to a conjecture of Oort (see [11], §5), Shimura families of curves $f: S \rightarrow C$ have bounded genus $g$.

For non-semistable families $f: S \rightarrow \mathbb{P}^{1}$ of curves of genus $g \geqslant 1$, it is well known that $s \geqslant 2$. If $f$ is non-isotrivial, Beauville proves that $s \geqslant 3$ (see [1]). Furthermore, for any genus $g \geqslant 2$, he constructed an example of such family with 3 singular fibers. There are indeed many families with $s=3$. In [12], U. Schmickler Hirzebruch classified all such fibrations with $g=1, s=2$ or 3 .

[^0]Non-isotrivial families of curves of genus $g \geqslant 2$ over $\mathbb{P}^{1}$ with 3 singular fibers have interesting arithmetic and geometric properties. In fact, they are isomorphic to some families defined over algebraic number fields. This is quite similar to Belyi's famous theorem that an algebraic curve is isomorphic to a curve defined over a number field if the curve is a finite cover of $\mathbb{P}^{1}$ ramified over 3 points (see [3]). Families with some extremal properties can usually be constructed from families with 3 singular fibers (see [6]).

We denote by $s_{1}$ the number of semistable singular fibers of $f: S \rightarrow \mathbb{P}^{1}$. If $s=3$ and $s_{1}=2$, Nguyen in [10] proves that $S$ is simply connected and $p_{g}(S)=q(S)=0$. He tends to believe that there are no families of curves of genus $g \geqslant 2$ with $s=3$ and $s_{1}=2$.

The main purpose of this note is to try to understand the geometric and arithmetic structure of families of curves $f: S \rightarrow \mathbb{P}^{1}$ of genus $g \geqslant 2$ with $s=3$ and $s_{1}=2$.

Theorem 1. Let $f: S \rightarrow \mathbb{P}^{1}$ be a relatively minimal family of curves of genus $g$ with 3 singular fibers. If two of the singular fibers are semistable, then $S$ is a rational surface, the Mordell-Weil group of $f$ is finite, the two semistable fibers consist of rational curves (may be singular) as their components, and the normal crossing model of the non-semistable singular fiber is a tree of smooth rational curves. Furthermore, we have:

$$
2 g \leqslant-K_{S}^{2} \leqslant 4 g-4
$$

For any positive integer $n$, we will construct examples of such fibrations of genus $g=2^{n}$ such that $-K_{S}^{2}=4 g-4$. It is difficult to construct examples with small $-K_{S}^{2}$.

Let $f: S \rightarrow \mathbb{P}^{1}$ be a relatively minimal fibration of genus $g$ with two singular fibers $F_{1}$ and $F_{2}$. Similarly to the proof of Theorem 1, one can prove easily that $S$ is a ruled surface, and:

$$
g\left(F_{1}\right)=g\left(F_{2}\right)=q(S)
$$

where $g\left(F_{i}\right)$ is the geometric genus of $F_{i}$, i.e., the sum of the genera of the normalization of the components in $F_{i}$. (See Remark 2.1.) But $q(S)$ is not necessarily zero as Nguyen expected in [10]. For example, the curves $C_{t}$ defined by $y^{2}=x^{6}+t^{3}$ induce a family $f: S \rightarrow \mathbb{P}^{1}$ of curves of genus 2 with two singular fibers at 0 and $\infty$, each singular fiber containing a smooth elliptic curve. Thus $q(S)=1$ (see Example 2 in Section 3 for details).

## 2. Proof of Theorem 1

For a relatively minimal fibration $f: S \rightarrow C$ of genus $g$ over a smooth curve $C$ of genus $b$, it is convenient to use the relative numerical invariants of the fibration:

$$
\begin{aligned}
& K_{f}^{2}=c_{1}^{2}(S)-8(g-1)(b-1) \\
& e_{f}=c_{2}(S)-4(g-1)(b-1) \\
& \chi_{f}=\chi\left(\mathcal{O}_{S}\right)-(g-1)(b-1) \\
& q_{f}=q(S)-b
\end{aligned}
$$

Let $F_{1}, \ldots, F_{s}$ be all singular fibers of $f$ and $l_{i}$ be the number of irreducible components of $F_{i}$. The rank of the MordellWeil group of $f$ is denoted by $r$. We have a formula to compute the rank $r$ (see [13], Theorem 3):

$$
r=\rho(S)-2-\sum_{i}\left(l_{i}-1\right)
$$

where $\rho(S)=\operatorname{rank} \operatorname{NS}(S)$ is the Picard number of $S$. Because the Mordell-Weil group is a finitely generated group, $r=0$ implies that the group is finite.

For a singular fiber $F$, we denote by $g(F)$ the sum of the geometric genus of its components. We denote by $\bar{F}=\sigma^{*} F$ the normal crossing model of $F$, i.e., $\sigma$ is the blowing-ups of the singular points of $F$ such that $\bar{F}=\sigma^{*} F$ is a normal crossing divisor. $N_{\bar{F}}:=g-p_{a}\left(\bar{F}_{\text {red }}\right)$, we have:

$$
0 \leqslant N_{\bar{F}} \leqslant g
$$

Note that $N_{\bar{F}}=g$, i.e., $p_{a}\left(\bar{F}_{\text {red }}\right)=0$, if and only if $\bar{F}$ is a tree of smooth rational curves. If $F$ is semistable, then $F=\bar{F}$ and $N_{F}=0$. The relative invariants can be computed respectively by using the modular invariants $\kappa(f), \lambda(f)$ and $\delta(f)$ :

$$
\left\{\begin{array}{l}
K_{f}^{2}=\kappa(f)+\sum_{i=1}^{s} c_{1}^{2}\left(F_{i}\right)  \tag{1}\\
e_{f}=\delta(f)+\sum_{i=1}^{s} c_{2}\left(F_{i}\right) \\
\chi_{f}=\lambda(f)+\sum_{i=1}^{s} \chi_{F_{i}}
\end{array}\right.
$$

where $c_{1}^{2}(F), c_{2}(F)$ and $\chi_{F}$ are the Chern numbers of the singular fiber $F$, which are non-negative rational numbers. When $g \geqslant 2$, one of them vanishes if and only if all the three vanish and this occurs if and only if $F$ is semistable (see [14,16] or [7]). So, for a semistable fibration $f$,

$$
K_{f}^{2}=\kappa(f), \quad e_{f}=\delta(f), \quad \chi_{f}=\lambda(f)
$$

The following formula, due to $\mathrm{Lu}, \mathrm{Tan}, \mathrm{Yu}$ and Zuo , is the main tool in our proof of Theorem 1:

Theorem 2. (See [8].) Let $s_{0}$ be the number of singular fibers satisfying $g(F)<g$. With the notations as above, we have:

$$
\begin{equation*}
2 \chi_{f}=\left(g-q_{f}\right)\left(2 b-2+s_{0}\right)-\sum_{i=1}^{s_{0}}\left(g\left(F_{i}\right)-q_{f}\right)-\left(h^{1,1}(S)-2 q_{f} b-2-\sum_{i=1}^{s}\left(l_{i}-1\right)\right)+\sum_{i=1}^{s_{0}} N_{\bar{F}_{i}} . \tag{2}
\end{equation*}
$$

Beauville in [1] proves that $g\left(F_{i}\right) \geqslant q_{f}$. So:

$$
\begin{equation*}
\mathcal{A}:=\sum_{i=1}^{s_{0}}\left(g\left(F_{i}\right)-q_{f}\right) \geqslant 0 \tag{3}
\end{equation*}
$$

The non-negativity of the second part is proved in [8]:

$$
\begin{equation*}
\mathcal{B}:=h^{1,1}(S)-2 q_{f} b-2-\sum_{i=1}^{s}\left(l_{i}-1\right) \geqslant 0 \tag{4}
\end{equation*}
$$

Note that $\rho(S) \leqslant h^{1,1}(S)$. For families $f: S \rightarrow \mathbb{P}^{1}$, we have $b=0$, so $r \leqslant \mathcal{B}$.
Lemma 2.1. Let $f: S \rightarrow \mathbb{P}^{1}$ be a relatively minimal fibration of genus $g$ with two semistable singular fibers $F_{1}$ and $F_{2}$, and one non-semistable fiber $F_{3}$. Then $\kappa(S)=-\infty$, and $4 g-4 \leqslant K_{f}^{2} \leqslant 6 g-8$.

Proof. The inequality $K_{f}^{2} \geqslant 4 g-4$ is proved in [18]. The inequality $K_{f}^{2}<6 g-6$ implies that $\kappa(S)=-\infty$ (see also [18]). So we only need to prove that $K_{f}^{2} \leqslant 6 g-8$.

We consider the semistable reduction $\tilde{f}: \tilde{S} \rightarrow \mathbb{P}^{1}$ of $f: S \rightarrow \mathbb{P}^{1}$, where the base change $\pi: \mathbb{P}^{1} \rightarrow \mathbb{P}^{1}$ is a cyclic cover totally ramified over $f\left(F_{2}\right)$ and $f\left(F_{3}\right)$.


Then number $\tilde{s}$ of the singular fibers of $\tilde{f}$ is at most $d+2$. By the strict canonical class inequality for $\tilde{f}: \tilde{S} \rightarrow \tilde{C}=\mathbb{P}^{1}$ (see [15,5,9]), we have:

$$
K_{\tilde{f}}^{2}<(2 g-2)(2 g(\tilde{C})-2+\tilde{s}) \leqslant 2 d(g-1)
$$

So $\kappa(f)=\frac{1}{d} \kappa(\tilde{f})=\frac{1}{d} K_{\tilde{f}}^{2}<2 g-2$.
If $c_{1}^{2}\left(F_{3}\right) \leqslant 4 g-5$, then we have:

$$
\begin{equation*}
K_{f}^{2}=\kappa(f)+c_{1}^{2}\left(F_{3}\right)<2 g-2+4 g-5=6 g-7 \tag{5}
\end{equation*}
$$

hence $K_{f}^{2} \leqslant 6 g-8$.
If $c_{1}^{2}\left(F_{3}\right)>4 g-5$, then by [7] (see also [17], Theorem 2.3), we have $g=2, c_{1}^{2}\left(F_{3}\right)=3.2$ and the dual graph of $F_{3}$ is as follows:


Here $\circ$ is a ( -2 )-curve and $\bullet$ is a ( -3 )-curve. The number is the multiplicity of the component in $F_{3}$.

In the process of the semistable reduction as above, we can take $d=10$. Then we see that the semistable reduction $\tilde{F}$ of $F$ is smooth (see [20]). Hence $\tilde{f}$ has exactly $d+1$ singular fibers, i.e., $\tilde{s}=d+1=11$. By the strict canonical class inequality, we have:

$$
K_{\tilde{f}}^{2}<(2 g-2)(2 g(\tilde{C})-2+\tilde{s})=(2 g-2)(d-1)=18,
$$

i.e., $K_{\tilde{f}}^{2} \leqslant 17$. So $\kappa(f) \leqslant \frac{1}{10} \cdot 17=1.7$. Hence $K_{f}^{2}=\kappa(f)+c_{1}^{2}\left(F_{3}\right) \leqslant 1.7+3.2=4.9$, i.e., $K_{f}^{2} \leqslant 4=6 g-8$. This completes the proof.

Lemma 2.2. With the notations and assumptions as above, the Mordell-Weil group of $f: S \rightarrow \mathbb{P}^{1}$ is finite, i.e.,

$$
\begin{equation*}
r=\rho(S)-2-\sum_{i}\left(l_{i}-1\right)=0 . \tag{6}
\end{equation*}
$$

$g\left(F_{i}\right)=q(S)$ for $i=1,2,3$ and $p_{a}\left(\bar{F}_{3, \text { red }}\right)=q(S)$.
Proof. Because $S$ is a ruled surface, $p_{g}(S)=0$. We have $\chi_{f}=g-q(S)$.
By (2),

$$
\chi_{f}=\frac{1}{2}(g-q(S))\left(-2+s_{0}\right)-\frac{1}{2} \mathcal{A}-\frac{1}{2} \mathcal{B}+\frac{1}{2} \mathcal{C},
$$

where:

$$
\mathcal{A}=\sum_{i=1}^{s_{0}}\left(g\left(\bar{F}_{i}\right)-q(S)\right) \geqslant 0, \quad \mathcal{B}=h^{1,1}(S)-2-\sum_{i=1}^{s}\left(l_{i}-1\right) \geqslant 0, \quad \mathcal{C}=N_{\bar{F}_{3}} \geqslant 0 .
$$

Because $p_{a}\left(\bar{F}_{3, \text { red }}\right) \geqslant g\left(\bar{F}_{3}\right) \geqslant q_{f}=q(S)$, we have:

$$
\mathcal{C}=N_{\bar{F}_{3}}=g-p_{a}\left(\bar{F}_{3}\right) \leqslant g-q(S) .
$$

Note that $s_{0} \leqslant s=3$. We have:

$$
g-q(S)=\chi_{f} \leqslant \frac{g-q(S)}{2}-\frac{1}{2} \mathcal{A}-\frac{1}{2} \mathcal{B}+\frac{1}{2} \mathcal{C} \leqslant g-q(S),
$$

which implies that $s_{0}=s=3, \mathcal{A}=\mathcal{B}=0$, and $\mathcal{C}=g-q(S)$. From $\mathcal{A}=0$, we see that $g\left(\bar{F}_{i}\right)=q(S)$. From $r \leqslant \mathcal{B}=0$, we get $r=0$. From $\mathcal{C}=g-q(S)$, we get $p_{a}\left(\bar{F}_{3, \text { red }}\right)=q(S)$.

Lemma 2.3. $($ See $[10]) ~ q.(S)=0$.
Proof. (Beauville) Suppose $q(S)>0$ and $\alpha: S \rightarrow \operatorname{Alb}(S)$ is the Albanese map. Because $S$ is a ruled surface, $\alpha$ is a fibration over a smooth curve $B$ of genus $q(S)$. The three singular fibers cannot be contracted by $\alpha$, so in each singular fiber $F_{i}$, there is a component $C_{i}$ which maps to $B$ surjectively. Denote by $\tilde{C}_{i}$ the normalization of $C_{i}$. Then $g\left(\tilde{C}_{i}\right) \geqslant q(S)$. On the other hand, $g\left(F_{i}\right)=q(S)$, so $g\left(C_{i}\right)=q(S)$ and the other components are rational curves (maybe singular).

If $q(S)>1$, then $\alpha: \tilde{C}_{i} \rightarrow B$ is an isomorphism. Hence $C_{1}$ is a section of $\alpha$, i.e., $C_{1} F_{b}=1$ for a general fiber $F_{b}$ of $\alpha$. The other components of $F_{1}$ are rational curves, which must be contracted by $\alpha$. Hence $F_{1} F_{b}=C_{1} F_{b}=1$, and the smooth fibers of $f$ are isomorphic to $B$, this is a contradiction because $f$ is not isotrivial.

If $q(S)=1$, similar to the proof above, we can assume that $\pi: \tilde{C}_{1} \rightarrow B$ induced by $\alpha$ is an unramified finite cover of degree $d>1$. We consider the pullback fibration $\tilde{\alpha}: \tilde{S}=S \times{ }_{B} \tilde{C}_{1} \rightarrow \tilde{C}_{1}$ of $\alpha$ under the base change $\pi$.


Because $\Pi^{-1}\left(C_{1}\right)$ contains a section of $\tilde{\alpha}$, and $g\left(\tilde{C}_{1}\right)=q(S)=1$, we know that the geometric genus of $\Pi^{-1}\left(C_{1}\right)$ is at least 2 . Let $\tilde{f}=\Pi \circ f: \tilde{S} \rightarrow \mathbb{P}^{1}$ and $\tilde{F}_{i}=\Pi^{*}\left(F_{i}\right)$. Then $g\left(\tilde{F}_{1}\right) \geqslant 2$. Note that only one singular fiber $\tilde{F}_{3}$ of $\tilde{f}$ is not semistable.

We claim that $\tilde{f}$ has connected fibers. Otherwise, we consider the Stein factorization,



Fig. 1.
$(0,0)$

$t=1$
$t=\infty$

Fig. 2.
Because only one singular fiber $\tilde{F}_{\tilde{\sim}}$ of $\tilde{f}$ may have multiple components, we see that $\varphi$ is a finite cover of degree $\operatorname{deg} \varphi>1$ ramified at worst over one point $\tilde{f}(\tilde{F})$ on $\mathbb{P}^{1}$. Such a finite cover $\varphi$ does not exist. Thus $\tilde{f}$ has connected fibers.

Now we know that $\tilde{f}$ is a fibration with 3 singular fibers $\tilde{F}_{1}, \tilde{F}_{2}$ and $\tilde{F}_{3}$, two fibers $\tilde{F}_{1}$ and $\tilde{F}_{2}$ are semistable fibers. From the discussion above, we get:

$$
g\left(\tilde{F}_{i}\right)=q(\tilde{S}) \leqslant 1,
$$

which contradicts to $g\left(\tilde{F}_{i}\right)=q(\tilde{S}) \geqslant 2$.
Finally, from Lemma 2.2, $p_{a}\left(\bar{F}_{3, \text { red }}\right)=0$, thus $\bar{F}_{3}$ is a tree of smooth rational curves.
Remark 2.1. Suppose $f: S \rightarrow \mathbb{P}^{1}$ is a relatively minimal fibration of genus $g \geqslant 2$ with two singular fibers $F_{1}$ and $F_{2}$. Similar to the proof above, we see that $S$ is a ruled surface, and

$$
s_{0}=2, \quad \mathcal{A}=\mathcal{B}=0, \quad p_{a}\left(\bar{F}_{i, \text { red }}\right)=g\left(\bar{F}_{i}\right)=q(S), \quad i=1,2 .
$$

We have seen in the introduction that $q(S)$ is not necessarily zero.

## 3. Examples

Example 1. Fibrations $f: S \rightarrow \mathbb{P}^{1}$ with $g=2^{n}, s=3$ and $s_{1}=2$, where $n=0,1,2, \ldots$.
Let $C_{0}$ be a curve in $\mathbb{P}^{1} \times \mathbb{P}^{1}$ defined by $t-x^{2}=0$. Let $L_{a}$ be the horizontal line defined by $x=a$, and $T_{b}$ be the vertical line defined by $t=b$. (See Fig. 1.)
$C_{0}$ meets the fiber $T_{1}$ transversely at two points. Choose one point $\left(x_{0}, 1\right) \in C_{0}$. We consider a double cover $\pi_{1}: P_{1}=$ $\mathbb{P}^{1} \times \mathbb{P}^{1} \rightarrow P_{0}=\mathbb{P}^{1} \times \mathbb{P}^{1}$ ramified over two horizontal lines $x=0$ and $x=x_{0}$. Let $C_{1}=\pi_{1}^{*} C_{0} \subset P_{1}$. Then $\#\left(C_{1} \cap T_{0}\right)=1$, $\#\left(C_{1} \cap T_{1}\right)=3$, and $\#\left(C_{1} \cap T_{\infty}\right)=2$. $C_{1}$ is a curve in $P_{1}$ as follows. (See Fig. 2.)
$C_{1}$ meets $T_{1}$ transversely at two points. Choose one point $\left(x_{1}, 1\right) \in C_{1}$. We consider the double cover $\pi_{2}: P_{2}=\mathbb{P}^{1} \times \mathbb{P}^{1} \rightarrow$ $P_{1}=\mathbb{P}^{1} \times \mathbb{P}^{1}$ ramified over two horizontal lines $x=0$ and $x=x_{1}$. Let $C_{2}=\pi_{2}^{*} C_{1}$. Then $\#\left(C_{2} \cap T_{0}\right)=1, \#\left(C_{2} \cap T_{1}\right)=5$, and $\#\left(C_{1} \cap T_{\infty}\right)=4$. $C_{2}$ meets $T_{1}$ transversely at two points.

Repeating this process, we get a curve $C_{n}$ of type $\left(2^{n+1}, 1\right)$ in $P_{n}=\mathbb{P}^{1} \times \mathbb{P}^{1}$ such that:

$$
\begin{aligned}
& \#\left(C_{n} \cap T_{0}\right)=1, \quad \#\left(C_{n} \cap T_{1}\right)=2^{n}+1, \quad \#\left(C_{n} \cap T_{\infty}\right)=2^{n}, \\
& P_{n} \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_{2} \rightarrow P_{1} \rightarrow P_{0} .
\end{aligned}
$$

Let

$$
B_{n}=C_{n}+T_{0}+L_{0}+L_{x_{n}} \equiv\left(2^{n+1}+2,2\right)
$$

$B_{n}$ is an even divisor with only ADE singularities. Let $\Pi_{n}: \Sigma_{n} \rightarrow \mathbb{P}^{1} \times \mathbb{P}^{1}$ be the double cover branched along $B_{n}$, and let $S_{n}$ be the minimal resolution of singularities of $\Sigma_{n}$.


The second projection $\mathbb{P}^{1} \times \mathbb{P}^{1} \rightarrow \mathbb{P}^{1}$ induces fibration $f_{n}: S_{n} \rightarrow \mathbb{P}^{1}$ of genus $g=2^{n}$ with $s=3$ singular fibers $F_{0}, F_{1}$ and $F_{\infty}$, and $F_{1}$ and $F_{\infty}$ are semistable. By a direct computation, we see that $S_{n}$ is rational and:

$$
-K_{S_{n}}^{2}=4 g-4
$$

Example 2. We consider the family of curves $C_{t}$ of genus 2 defined by $y^{2}=x^{6}+t^{3}$ for $t \in \mathbb{P}^{1}$. We consider $\mathbb{P}^{1} \times \mathbb{P}^{1}$ as the compactification of $\mathbb{C} \times \mathbb{C}$ with coordinate $(x, t)$. Let $X \rightarrow \mathbb{P}^{1} \times \mathbb{P}^{1}$ be a double cover ramified over a curve defined by
$x^{6}+t^{3}=0$ and $\mathbb{P}^{1} \times\{\infty\}$. Let $S$ be the minimal resolution of the two singularities of $X$ over $(0,0)$ and $(\infty, \infty)$. Then we get a fibration $f: S \rightarrow \mathbb{P}^{1}$ of curves of genus 2 with two singular fibers at $t=0$ and $\infty$. The two singular fibers have the same type. Here is the canonical resolution.

The fiber $F_{0}$ at $t=0$.

$x^{6}+t^{3}=0$
$\xrightarrow{\sigma_{3}}$
$\left(\bar{E}_{1},-1\right)$

$F_{0}$

The fiber $F_{\infty}$ at $t=\infty$. The singular point of the branch locus is at $(\infty, \infty)$. Choose the affine coordinates $u=\frac{1}{x}=0$ and $s=\frac{1}{t}=0$. Then the singular point is at $(0,0)$.

where $\sigma_{i}$ and $\sigma_{i}^{\prime}$ are blowing-ups and blowing-downs, and $\pi$ and $\pi^{\prime}$ are the double cover. From the computation above, each singular fiber contains a smooth elliptic curve $\bar{E}_{2}$. Thus $q(S)=1$.

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