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# Semisimple Lie groups satisfy property RD, a short proof





## Les groupes de Lie semi-simples ont la propriété RD, une preuve courte

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ARTICLE INFO	ABSTRACT
Article history: Received 6 December 2012 Accepted after revision 21 May 2013 Available online 30 May 2013 Presented by the Editorial Board	We give a short elementary proof of the fact that connected semisimple real Lie groups satisfy property RD. The proof is based on a process of linearisation. © 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved. R É S U M É
	Nous donnons dans cette note une preuve courte et élémentaire du fait que les groupes de Lie semi-simples réels connexes satisfont la propriété RD. La preuve est basée sur un procédé de linéarisation. © 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

## 1. Introduction

A length function  $L: G \to \mathbb{R}^*_+$  on a locally compact group G is a measurable function satisfying:

(i) L(e) = 0 where *e* is the neutral element of *G*,

(ii)  $L(g^{-1}) = L(g)$ ,

(iii)  $L(gh) \leq L(g) + L(h)$ .

A unitary representation  $\pi : G \to U(H)$  on a complex Hilbert space has property RD with respect to *L* if there exists C > 0 and  $d \ge 1$  such that for each pair of unit vectors  $\xi$  and  $\eta$  in *H*, we have:

$$\int_{G} \frac{|\langle \pi(g)\xi,\eta\rangle|^2}{(1+L(g))^d} \,\mathrm{d}g \leqslant C$$

where dg is a (left) Haar measure on G, see [10]. We say that G has property RD if its regular representation has property RD with respect to L. First established for free groups by U. Haagerup in [6], property RD has been introduced and studied as such by P. Jolissaint in [8], who notably established it for groups of polynomial growth, and for classical hyperbolic groups. See [12, Chap. 8, p. 69] for more details.

If  $\pi$  denotes a unitary representation on a Hilbert space H, then  $\overline{\pi}$  denotes its conjugate representation on the conjugate Hilbert space  $\overline{H}$ . The process of linearisation consists in working with  $\sigma : G \to U(\overline{H} \otimes H)$  the unitary representation  $\sigma = \overline{\pi} \otimes \pi$ , see [3, Section 2.2].

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A connected semisimple real Lie group with finite center can be written G = KP, where K is a compact connected subgroup, and P a closed amenable subgroup. We denote by  $\Delta_P$  the right-modular function of P. Extend to G the map  $\Delta_P$  of P as  $\Delta : G \to \mathbb{R}^*_+$  with  $\Delta(g) = \Delta(kp) := \Delta_P(p)$ . It is well defined because  $K \cap P$  is compact (observe that  $\Delta_{P|K\cap P} = 1$ ). The quotient G/P carries a unique quasi-invariant measure  $\mu$ , such that the Radon–Nikodym derivative at  $(g, x) \in G \times G/P$ , denoted by  $c(g, x) = \frac{dg_*\mu}{d\mu}(x)$ , with  $g_*\mu(A) = \mu(g^{-1}A)$ , satisfies  $\frac{dg_*\mu}{d\mu}(x) = \frac{\Delta(gx)}{\Delta(x)}$  for all  $g \in G$  and  $x \in G/P$  (notice that for all  $g \in G$ , the function  $x \in G/P \mapsto \frac{\Delta(gx)}{\Delta(x)} \in \mathbb{R}^*_+$  is well defined). We refer to [2, Appendix B, Lemma B.1.3, pp. 344–345] for more details. Consider the quasi-regular representation  $\lambda_{G/P} : G \to U(L^2(G/P))$  associated with P, defined by  $(\lambda_{G/P}(g)\xi)(x) = c(g^{-1}, x)^{\frac{1}{2}}\xi(g^{-1}x)$ . Denote by dk the Haar measure on K, and under the identification  $G/P = K/(K \cap P)$ , denote by d[k] the measure  $\mu$  on G/P.

The well-known Harish-Chandra function is defined by  $\Xi(g) := \langle \lambda_{G/P}(g) \mathbf{1}_{G/P}, \mathbf{1}_{G/P} \rangle$  where  $\mathbf{1}_{G/P}$  denotes the characteristic function of the space G/P.

In the rest of the paper, we set  $\sigma = \overline{\lambda_{G/P}} \otimes \lambda_{G/P}$ . Observe that  $\overline{L^2(G/P)} \otimes L^2(G/P) \cong L^2(G/P \times G/P)$ , via:  $\xi \otimes \eta \mapsto ((x, y) \mapsto \overline{\xi(x)}\eta(y))$ . Notice that  $\sigma$  preserves the cone of positive functions on  $L^2(G/P \times G/P)$ .

Let *G* be a (non-compact) connected semisimple real Lie group. Let  $\mathfrak{g}$  be its Lie algebra. Let  $\theta$  be a Cartan involution. Define the bilinear form denoted by (X, Y) such that for all  $X, Y \in \mathfrak{g}$ ,  $(X, Y) = -B(X, \theta(Y))$ , where *B* is the Killing form. Set  $|X| = \sqrt{(X, X)}$ . Write  $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$  the eigenvector space decomposition associated with  $\theta$  ( $\mathfrak{l}$  for the eigenvalue 1). Let *K* be the compact subgroup defined as the connected subgroup whose Lie algebra  $\mathfrak{l}$  is the set of fixed points of  $\theta$ . Fix  $\mathfrak{a} \subset \mathfrak{p}$  a maximal Abelian subalgebra of  $\mathfrak{p}$ . Consider the roots system  $\Sigma$  associated with  $\mathfrak{a}$  and let  $\Sigma^+$  be the set of positive roots, and define the corresponding positive Weyl chamber as:

$$\mathfrak{a}^+ := \{ H \in \mathfrak{a}, \ \alpha(H) > 0, \ \forall \alpha \in \Sigma^+ \}.$$

Let  $A^+ = Cl(\exp(\mathfrak{a}^+))$ , where *Cl* denotes the closure of  $\exp(\mathfrak{a}^+)$ . Consider the corresponding polar decomposition  $KA^+K$ . Then define the length function:

$$L(g) = L(k_1 e^H k_2) := |H|$$

where  $g = k_1 e^H k_2$  with  $e^H \in A^+$ . Notice that *L* is *K* bi-invariant. The desintegration of the Haar measure on *G* according to the polar decomposition is:

$$dg = dk J(H) dH dk$$

where dk is the Haar measure on K, dH the Lebesgue measure on  $a^+$ , and

$$J(H) = \prod_{\alpha \in \Sigma^+} \left( \frac{e^{\alpha(H)} - e^{-\alpha(H)}}{2} \right)^{n_{\alpha}}$$

where  $n_{\alpha}$  denotes the dimension of the root space associated with  $\alpha$ . See [9, Chap. V, Section 5, Proposition 5.28, pp. 141–142], [5, Chap. 2, §2.2, p. 65] and [5, Chap. 2, Proposition 2.4.6, p. 73] for more details.

The aim of this note is to give a short proof of the following known result [4,7].

**Theorem 1** (C. Herz). Let G be a connected real semisimple Lie group with finite center. Then G has property RD with respect to L.

See [4, Proposition 5.5 and Lemma 6.3] for the case *G* has infinite center.

### 2. Proof

**Proof.** We shall prove that the quasi-regular representation has property RD with respect to *L* defined above. This implies that the regular representation has property RD with respect to *L* by Lemma 2.3 in [11]. Write G = KP, where *K* is a compact subgroup and *P* is a closed amenable subgroup of *G*. It is sufficient to prove that there exists  $d_0 \ge 1$  and  $C_0 \ge 0$  such that  $\int_G \frac{(\lambda_G/P(g)\xi,\xi)^2}{(1+L(g))^{d_0}} dg < C_0$ , for positive functions  $\xi$ , with  $\|\xi\| = 1$ .

Take  $\xi \in L^2(G/P)$  such that  $\xi \ge 0$ , and  $\|\xi\| = 1$ . Define the function:

$$F: G/P \times G/P \to \mathbb{R}_+,$$
  
(x, y)  $\mapsto \int_K \sigma(k)(\xi \otimes \xi)(x, y) \, \mathrm{d}k.$ 

For all  $(x, y) \in G/P \times G/P$ , we have by the Cauchy–Schwarz inequality:

$$\int_{K} \sigma(k)(\xi \otimes \xi)(x, y) \, \mathrm{d}k = \int_{K} \xi(k^{-1}x)\xi(k^{-1}y) \, \mathrm{d}k$$
$$\leqslant \left(\int_{K} \xi^{2}(k^{-1}x) \, \mathrm{d}k\right)^{\frac{1}{2}} \left(\int_{K} \xi^{2}(k^{-1}y) \, \mathrm{d}k\right)^{\frac{1}{2}}.$$

Observe that the function  $f : x \in G/P \mapsto \int_K \xi^2 (k^{-1}x) \, dk \in \mathbb{R}_+$  is constant. Indeed, fix  $x \in G/P$  and let y in G/P. Write y = hx for some  $h \in K$  (as K acts transitively on G/P). By invariance of the Haar measure, we have  $f(y) = \int_K \xi^2 (k^{-1}y) \, dk = \int_K \xi^2 (k^{-1}hx) \, dk = \int_K \xi^2 (k^{-1}x) \, dk = f(x)$ . If e is the neutral element in G, we write  $[e] \in G/P$ . We have, for all  $x \in G/P$ , f(x) = f([e]).

Hence, for all  $x \in G/P$  we have:

$$\int_{K} \xi^{2}(k^{-1}x) dk = \int_{K} \xi^{2}(k^{-1}[e]) dk$$
$$= \int_{K/K \cap P} \xi^{2}([k^{-1}]) d[k]$$
$$= \|\xi\|^{2} = 1.$$

Therefore  $||F||_{\infty} := \sup\{F(x, y), (x, y) \in G/P \times G/P\} \leq 1$ . Hence  $0 \leq F \leq 1_{G/P \times G/P}$ , where  $1_{G/P \times G/P}$  denotes the characteristic function of  $G/P \times G/P$ .

Let *r* be the number of indivisible positive roots in a. We know that there exists C > 0 such that, for all  $H \in \mathfrak{a}$  where  $e^H \in A^+$ , we have:

$$\Xi(e^{H}) \leqslant C e^{-\rho(H)} (1 + L(e^{H}))^{t}$$

with  $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} n_\alpha \alpha \in \mathfrak{a}^+$ , see [5, Chap. 4, Theorem 4.6.4, p. 161]. Hence for  $d_0 > \dim(\mathfrak{a}) + 2r$ , we have:

$$\int\limits_{\mathfrak{a}^+} \frac{\Xi^2(e^H)}{(1+L(e^H))^{d_0}} J(H) \, \mathrm{d} H < \infty.$$

We obtain for all  $d \ge 0$  and for all positive functions  $\xi$ , with  $\|\xi\| = 1$ :

$$\begin{split} \int_{G} \frac{\langle \lambda_{G/P}(g)\xi,\xi \rangle^{2}}{(1+L(g))^{d}} \, \mathrm{d}g &= \int_{G} \frac{\langle \lambda_{G/P}(g)\xi,\xi \rangle \langle \lambda_{G/P}(g)\xi,\xi \rangle}{(1+L(g))^{d}} \, \mathrm{d}g \\ &= \int_{G} \frac{\langle \sigma(g)\xi \otimes \xi,\xi \otimes \xi \rangle}{(1+L(g))^{d}} \, \mathrm{d}g \\ &= \int_{K} \int_{\mathfrak{a}^{+}} \int_{K} \frac{\langle \sigma(k_{1}e^{H}k_{2})\xi \otimes \xi,\xi \otimes \xi \rangle}{(1+L(k_{1}e^{H}k_{2}))^{d}} J(H) \, \mathrm{d}k_{1} \, \mathrm{d}H \, \mathrm{d}k_{2} \\ &= \int_{K} \int_{\mathfrak{a}^{+}} \int_{K} \frac{\langle \sigma(e^{H})\sigma(k_{2})(\xi \otimes \xi), \sigma(k_{1}^{-1})(\xi \otimes \xi) \rangle}{(1+L(e^{H}))^{d}} J(H) \, \mathrm{d}k_{1} \, \mathrm{d}H \, \mathrm{d}k_{2} \\ &= \int_{\mathfrak{a}^{+}} \frac{\langle \sigma(e^{H})(\int_{K} \sigma(k_{2})(\xi \otimes \xi) \, \mathrm{d}k_{2}), (\int_{K} \sigma(k_{1}^{-1})(\xi \otimes \xi) \, \mathrm{d}k_{1}) \rangle}{(1+L(e^{H}))^{d}} J(H) \, \mathrm{d}H \\ &= \int_{\mathfrak{a}^{+}} \frac{\langle \sigma(e^{H})F, F \rangle}{(1+L(e^{H}))^{d}} J(H) \, \mathrm{d}H \\ &\leq \int_{\mathfrak{a}^{+}} \frac{\langle \sigma(e^{H})\mathbf{1}_{G/P \times G/P}, \mathbf{1}_{G/P \times G/P} \rangle}{(1+L(e^{H}))^{d}} J(H) \, \mathrm{d}H \end{split}$$

$$= \int_{\mathfrak{a}^{+}} \frac{\langle \lambda_{G/P}(e^{H}) \mathbf{1}_{G/P}, \mathbf{1}_{G/P} \rangle^{2}}{(1 + L(e^{H}))^{d}} J(H) \, \mathrm{d}H$$
$$= \int_{\mathfrak{a}^{+}} \frac{\Xi^{2}(e^{H})}{(1 + L(e^{H}))^{d}} J(H) \, \mathrm{d}H.$$

Take  $d_0 > \dim(\mathfrak{a}) + 2r$  and  $C_0 = \int_{\mathfrak{a}^+} \frac{\Xi^2(e^H)}{(1+L(e^H))^{d_0}} J(H) \, dH$ . We have found  $d_0 \ge 1$  and  $C_0 > 0$  such that for all positive functions  $\xi$  in  $L^2(G/P)$  with  $||\xi|| = 1$ , we have:

$$\int_{G} \frac{\langle \lambda_{G/P}(g)\xi,\xi \rangle^2}{(1+L(g))^{d_0}} \, \mathrm{d}g \leqslant C_0. \qquad \Box$$

**Remark 1.** The same approach applies to algebraic semisimple Lie groups over local fields. See [1, Section 1, (1.3)] and [13, Lemma II.1.5.].

**Remark 2.** It's not hard to see that this approach shows that the representations of the principal series of G (of class one, see [5, (3.1.12), p. 103]) satisfy also property RD.

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## References

- [1] J. Arthur, A local trace formula, Publ. Math. Inst. Hautes Études Sci. 73 (1991) 5–96.
- [2] B. Bekka, P. de la Harpe, A. Valette, Kazhdan's Property (T), New Math. Monogr., vol. 11, Cambridge University Press, Cambridge, 2008.
- [3] A. Boyer, Quasi-regular representations and property RD, preprint, arXiv:1305.0480, 2013.
- [4] I. Chatterji, C. Pittet, L. Saloff-Coste, Connected Lie groups and property RD, Duke Math. J. 137 (3) (2007) 511–536.
- [5] R. Gangolli, V.S. Varadarajan, Harmonic Analysis of Spherical Functions on Real Reductive Groups, Springer-Verlag, New York, 1988.
- [6] U. Haagerup, An example of a nonnuclear C\*-algebra which has the metric approximation property, Invent. Math. 50 (3) (1978/1979) 279–293.
- [7] C. Herz, Sur le phénomène de Kunze-Stein, C. R. Acad. Sci. Paris, Sér. A-B 271 (1970) A491-A493.
- [8] P. Jolissaint, Rapidly decreasing functions in reduced C\*-algebras of groups, Trans. Amer. Math. Soc. 317 (1) (1990) 167-196.
- [9] A.-W. Knapp, Representation Theory of Semisimple Groups, Princeton Landmarks Math., 2001.
- [10] M. Perrone, Rapid decay and weak containment of unitary representations, 2009, unpublished notes.
- [11] Y. Shalom, Rigidity, unitary representations of semisimple groups, and fundamental groups of manifolds with rank one transformation group, Ann. Math. (2) 152 (1) (2000) 113–182.
- [12] A. Valette, Introduction to the Baum-Connes Conjecture, Lectures Math. ETH Zürich, Birkhäuser Verlag, Basel, 2002.
- [13] J.-L. Walspurger, La formule de Plancherel pour les groupes p-adiques. D'après Harish-Chandra, J. Inst. Math. Jussieu 2 (2) (April 2003) 235-333.