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Option price with stochastic volatility for both fast and slow mean-reverting regimes



Prix de l'option à volatilité stochastique pour les régimes de retour à la moyenne, rapide ou lent

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ABSTRACT

The Heston model of stochastic volatility has been widely adopted in modern finance, especially in option pricing. Usually, the model can be classified as being in one of two different regimes: the fast mean-reverting regime and the slow mean-reverting regime. Different approximations are needed for each regime. We show a surprising result: the solution in both regimes can be approximated by an identical expression. The predictions of the approximation are in excellent agreement with the numerical solutions of the Heston model in both regimes.

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RÉSUMÉ

Le modèle de volatilité stochastique de Heston a été largement utilisé dans la théorie financière moderne, en particulier pour déterminer le prix des options. Habituellement, ce modèle peut prendre en compte deux régimes différents : le régime de retour rapide à la moyenne et celui de retour lent à la moyenne. Deux solutions différentes ont été données, selon le régime du modèle. Nous démontrons un résultat surprenant : les deux solutions peuvent être approchées par une formule identique. Dans chaque régime, les prédictions de l'approximation sont très proches des solutions numériques du modèle de Heston.

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The Heston model [4] assumes that the dynamics of the stock *s* and the volatility $\sqrt{v_t}$ are governed by

$$ds = \nu s dt + \sqrt{\nu_t} s dB_t^s, \qquad d\nu_t = \kappa (\theta - \nu_t) dt + \xi \sqrt{\nu_t} dB_t^\nu, \tag{1}$$

where ν is the rate of return of the stock, θ is the long-run average price volatility, κ is the rate at which ν_t reverts to θ , ξ is the volatility of the volatility (which determines the variance of ν_t), B_t^s and B_t^{ν} are the Brownian motions satisfying $dB_t^s dB_t^{\nu} = \rho dt$, and ρ is the correlation coefficient. Since one can introduce discounted financial instruments and a discounted factor e^{-qt} in stock price processes to eliminate the effect of the constant interest rate r and the constant dividend

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rate q, respectively, we set q and r to zero in this paper. A no-arbitrage argument shows that the option price f(t, s, v) satisfies:

$$f_t + \left[\kappa(\theta - \nu) - \lambda\xi\nu\right]f_\nu + \frac{1}{2}\nu s^2 f_{ss} + \rho\xi s\nu f_{s\nu} + \frac{1}{2}\xi^2\nu f_{\nu\nu} = 0$$
(2)

with the payoff $f(T, s, v) = (s - k)^+$ for a call and $f(T, s, v) = (k - s)^+$ for a put, where k is the strike price and $\lambda(t, s, v) = \lambda v^{\frac{1}{2}}$ is the price of volatility risk. Since making a change of variables $\bar{\kappa} = \kappa + \xi \lambda$ and $\bar{\theta} = \kappa \theta / \bar{\kappa}$ removes the effect of the price of volatility risk λ , we set $\lambda = 0$. Thus, Eq. (2) becomes:

$$f_t + \kappa (\theta - \nu) f_\nu + \frac{1}{2} \nu s^2 f_{ss} + \rho \xi s \nu f_{s\nu} + \frac{1}{2} \xi^2 \nu f_{\nu\nu} = 0.$$
(3)

The solution of Eq. (3) can be expressed as two integrals of complex function [4], and must be evaluated by numerical methods. Several authors showed that there are difficulties in evaluating this integral numerically; see [5,6]. Therefore, it is important to develop approximate solutions for the Heston model.

For a fast mean-reverting process [2], one replaces κ by $\frac{\kappa}{\epsilon}$ and ξ by $\frac{\xi}{\sqrt{\epsilon}}$ with $0 < \epsilon \ll 1$. The parameter ϵ represents the speed of mean reversion. Thus, Eq. (3) becomes:

$$f_t + \frac{\kappa}{\epsilon}(\theta - \nu)f_\nu + \frac{1}{2}\nu s^2 f_{ss} + \frac{\xi}{\sqrt{\epsilon}}s\rho\nu f_{s\nu} + \frac{1}{2}\frac{\xi^2}{\epsilon}\nu f_{\nu\nu} = 0.$$
(4)

In a slow mean-reverting regime [2], κ is replaced by $\delta \kappa$ and ξ by $\sqrt{\delta \xi}$ with $0 < \delta \ll 1$, and f satisfies:

$$f_t + \delta\kappa(\theta - \nu)f_\nu + \frac{1}{2}\nu s^2 f_{ss} + \sqrt{\delta\xi}\rho\nu s f_{s\nu} + \frac{1}{2}\delta\xi^2\nu f_{\nu\nu} = 0.$$
(5)

J. Han et al. [3] developed a closed-form approximate solution for the option prices in the fast mean-reverting regime. Their results for f are the following expansion in a power series of $\epsilon^{1/2}$:

$$f = f_0 + \sqrt{\epsilon} f_1 + \epsilon f_2 + \epsilon^{\frac{3}{2}} f_3 + \cdots,$$
(6)

where

$$f_0(s, z; \epsilon) = sN(d^+) - kN(d^-) \quad \text{for call, and} \quad f_0(s, z; \epsilon) = kN(-d^-) - sN(-d^+) \quad \text{for put,}$$
(7)

$$f_1(\tau, s, z; \epsilon) = g_1(\tau, z)G_1(s, z), \quad \text{for both call and put,}$$
(8)

$$f_2(\tau, s, z; \epsilon) = g_2(\tau, z)G_2(s, z) + h_2(\tau, z)H_2(s, z) + m_2(\tau, z)M_2(s, z)$$
 for both call and put. (9)

Here $\tau = T - t$, $z = \theta \tau + \frac{\epsilon}{\kappa} (1 - e^{-\kappa \tau/\epsilon})(v - \theta)$, $d^{\pm} = (\ln(s/k) \pm z/2)/\sqrt{z}$, N(x) is the *c.d.f.* of the standard normal distribution, and

$$\begin{split} G_{1}(s,z) &= -\frac{1}{2}\rho\frac{\xi}{\kappa}d^{-}z^{-1}k\phi(d^{-}), \qquad G_{2}(s,z) = \frac{1}{4}\rho^{2}\frac{\xi^{2}}{\kappa^{2}}k\phi(d^{-})z^{-\frac{5}{2}}\big[3-3(d^{-})^{2}-3d^{+}d^{-}+d^{+}(d^{-})^{3}\big], \\ H_{2}(s,z) &= -\frac{1}{2}\rho^{2}\frac{\xi^{2}}{\kappa^{2}}k\phi(d^{-})z^{-\frac{3}{2}}\big[1-(d^{-})^{2}\big], \qquad M_{2}(s,z) = \frac{1}{8}\frac{\xi^{2}}{\kappa^{2}}k\phi(d^{-})z^{-\frac{3}{2}}(d^{+}d^{-}-1), \\ g_{1}(\tau,z) &= A(\tau)z + \theta B(\tau), \qquad g_{2}(\tau,z) = \frac{1}{2}\big[A(\tau)z + \theta B(\tau)\big]^{2}, \\ h_{2}(\tau,z) &= C(\tau)z + \theta D(\tau), \qquad m_{2}(\tau,z) = E(\tau)z + \theta F(\tau), \end{split}$$

where $\phi(x)$ is the *p.d.f.* of the standard normal distribution, and

$$\begin{split} A(\tau) &= 1 - \frac{\kappa}{\epsilon} \frac{\tau e^{-\kappa\tau/\epsilon}}{1 - e^{-\kappa\tau/\epsilon}}, \qquad F(\tau) = \frac{1}{2} \frac{\epsilon}{\kappa} \left(1 - e^{-2\frac{\kappa\tau}{\epsilon}} \right) - 2\frac{\epsilon}{\kappa} \left(1 - e^{-\frac{\kappa\tau}{\epsilon}} \right) + \frac{\tau e^{-\kappa\tau/\epsilon}}{1 - e^{-\kappa\tau/\epsilon}} \left[2\frac{\kappa\tau}{\epsilon} - \left(1 - e^{-\frac{\kappa\tau}{\epsilon}} \right) \right], \\ B(\tau) &= \frac{\kappa}{\epsilon} \frac{\tau^2 e^{-\kappa\tau/\epsilon}}{1 - e^{-\kappa\tau/\epsilon}} - \frac{\epsilon}{\kappa} \left(1 - e^{-\frac{\kappa\tau}{\epsilon}} \right), \qquad D(\tau) = \tau e^{-\frac{\kappa\tau}{\epsilon}} - 2\frac{\epsilon}{\kappa} \left(1 - e^{-\frac{\kappa\tau}{\epsilon}} \right) + \frac{\tau e^{-\kappa\tau/\epsilon}}{1 - e^{-\kappa\tau/\epsilon}} \left[\frac{\kappa\tau}{\epsilon} + \frac{1}{2} \left(\frac{\kappa\tau}{\epsilon} \right)^2 \right], \\ C(\tau) &= 1 - \frac{e^{-\kappa\tau/\epsilon}}{1 - e^{-\kappa\tau/\epsilon}} \left[\frac{\kappa\tau}{\epsilon} + \frac{1}{2} \left(\frac{\kappa\tau}{\epsilon} \right)^2 \right], \qquad E(\tau) = 1 - \frac{e^{-\kappa\tau/\epsilon}}{1 - e^{-\kappa\tau/\epsilon}} \left[2\frac{\kappa\tau}{\epsilon} - \left(1 - e^{-\frac{\kappa\tau}{\epsilon}} \right) \right]. \end{split}$$

The approximation (6) is in excellent agreement with numerical solutions of the Heston model of stochastic volatility in the fast mean-reverting process [3]. Since the approximate solution is a series expansion in terms of ϵ , it is expected that the approximation will give very large errors for large ϵ . We plot the ratio of the option price to the stock price, f/s, versus the speed of mean reversion ϵ in the top row of Fig. 1. The solid curves are the numerical solution of the Heston model. The dashed curves are the second-order approximate solutions given by Eqs. (6)–(9) [3]. The relative errors between



Fig. 1. The ratio of the option price to the stock price, f/s, is plotted against the speed of mean reversion ϵ for the negatively correlated case ($\rho = -0.5$) and for uncorrelated case ($\rho = 0$) in the top row. The solid curves are the numerical solution of the Heston model. The dashed curves are the second-order approximate solutions given by Eqs. (6)–(9) [3]. The corresponding relative errors are plotted in the bottom row. The left, middle and right columns are for in-the-money case (k/s = 0.8), at-the-money case (k/s = 1) and out-of-the-money case (k/s = 1.2), respectively. The other parameters are: $\tau = 1$, $\kappa = 1$, $\nu = \theta = 0.04$, and $\xi = 0.15\sqrt{2}$.

Fig. 1. Le ratio du prix de l'option pour le prix de l'action, f/s, est tracé en fonction de la vitesse de retour à la moyenne ϵ pour le cas corrélé négativement ($\rho = -0.5$) et pour le cas non corrélé ($\rho = 0$) dans la rangée supérieure. Les courbes en trait plein constituent la solution numérique du modèle de Heston. Les courbes en pointillés sont les solutions approximatives de deuxième ordre données par les Éqs. (6)–(9) [3]. Les erreurs relatives correspondantes sont tracées dans la rangée inférieure. Les colonnes de gauche, au milieu et à droite sont le cas «in-the-money» (k/s = 0.8), le cas «at-the-money» (k/s = 1.2), respectivement. Les autres paramètres sont : $\tau = 1$, $\kappa = 1$, $\nu = \theta = 0.04$, et $\xi = 0.15\sqrt{2}$.

the approximate solutions and the exact numerical solutions are plotted in the bottom row of Fig. 1. Fig. 1 shows a very surprising result: the approximate solution is in excellent agreement with numerical solutions, not only for small ϵ , but also for large ϵ . This means that the approximation given by Eqs. (6)–(9) for the fast mean-reverting regime is also valid for the slow mean-reverting regime!

Now we prove this is true. The results are summarized in the following two theorems. Theorem 1 gives an approximate solution for the slow mean-reverting regime. Theorem 2 proves that the solutions given in Theorem 1 are identical to Eqs. (6)-(9), which was developed for the fast mean-reverting regime.

Theorem 1. In the slow mean-reverting regime, the option price \tilde{f} has the following expansion in a power series of $\delta^{1/2}$:

$$\tilde{f} = \tilde{f}_0 + \sqrt{\delta}\tilde{f}_1 + \delta\tilde{f}_2 + \cdots$$
(10)

with

$$\tilde{f}_0 = f_0(s, y; 1/\delta), \qquad \tilde{f}_1 = f_1(\tau, s, y; 1/\delta) \quad and \quad \tilde{f}_2 = f_2(\tau, s, y; 1/\delta),$$
(11)

where $y = \theta \tau + (v - \theta)(1 - e^{-\delta \kappa \tau})/(\delta \kappa)$ and the functional form of f_0 , f_1 and f_2 are given by Eqs. (7)–(9).

Proof. One usually applies the regular perturbation method in term of δ to Eq. (5) to develop an approximate solution for the slow mean-reverting regime. However, such an approach will lead to a secular effect [1], namely, as $v \to \infty$, the error of the approximate solution will tend to infinity as well. To resolve this problem, we use a multiple-scale approach [1]. Following this method, we introduce a new scale *y* given above, which enlarges the variables space from (t, s, v) to (τ, s, v, y) , and apply Taylor's Theorem in the enlarged space. Finally, we express *y* in terms of τ and *v*, and solve the resulting equations to obtain an expansion. Following this multiple-scale approach, Eq. (5) can be expressed as:

$$(\mathcal{L}_0 + \sqrt{\delta}\mathcal{L}_1 + \delta\mathcal{L}_2)f = 0 \tag{12}$$

with

$$\mathcal{L}_{0} = \frac{\partial}{\partial t} + \delta \kappa (\theta - \nu) \frac{\partial}{\partial \nu} + \frac{1}{2} \nu s^{2} \frac{\partial^{2}}{\partial s^{2}}, \qquad \mathcal{L}_{1} = \xi \rho \nu s \frac{\partial^{2}}{\partial s \partial \nu} \quad \text{and} \quad \mathcal{L}_{2} = \frac{1}{2} \xi^{2} \nu \frac{\partial^{2}}{\partial \nu^{2}}.$$

Substituting Eq. (10) into Eq. (12), we get a set of equations:

$$O\left(\delta^{\frac{n}{2}}\right): \quad \mathcal{L}_0 \tilde{f}_n + \mathcal{L}_1 \tilde{f}_{n-1} + \mathcal{L}_2 \tilde{f}_{n-2} = 0 \quad \text{with } \tilde{f}_n(0, s, z) = \delta_{0n} \cdot \text{payoff}, \tag{13}$$

where $f_{-1} = f_{-2} = 0$, and δ_{0n} is the Kronecker delta function, with $\delta_{0n} = 1$ for n = 0, and $\delta_{0n} = 0$ for $n \neq 0$. Although Eq. (13) looks extremely complicated, we are able to obtain closed-form expressions for the solutions of these equations, and the results are given by Eq. (11). \Box

We comment that the operator \mathcal{L}_0 contains the term $\delta \kappa (\theta - \nu) \frac{\partial}{\partial \nu}$. In regular perturbation, this term will be classified in \mathcal{L}_2 and, unfortunately, such a classification would lead to the secular effect. The singular perturbation method promotes this terms to \mathcal{L}_0 and consequently removes the secular effect.

Theorem 2. The approximate solution \tilde{f} for the slow mean-reverting regime given in Theorem 1 is identical to the approximation f given by Eqs. (6)–(9) for the fast mean-reverting regime, namely, $\tilde{f} = f$.

Proof. It is straightforward to check that, after replacing δ by $\frac{1}{\epsilon}$, Eq. (10) becomes Eq. (6) and Eq. (11) becomes Eqs. (7)–(9). \Box

Theorems 1 and 2 prove that although the fast mean-reverting regime and the slow mean-reverting regime are very different, the solution given by Eqs. (6)–(9) gives an accurate approximation for both regimes. This explains the surprising phenomenon shown in Fig. 1.

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