

Group Theory

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I





L^2 -Betti numbers of locally compact groups



Nombres L² de Betti des groupes localement compacts

Henrik Densing Petersen¹

SB-MATHGEOM-EGG, EPFL, Station 8, CH-1015, Lausanne, Switzerland

ARTICLE INFO

Article history: Received 20 March 2013 Accepted 28 May 2013 Available online 12 June 2013

Presented by the Editorial Board

ABSTRACT

The present paper is a summary and overview of results obtained in the author's thesis, " L^2 -Betti Numbers of Locally Compact Groups", wherein the definition of L^2 -Betti numbers for countable groups is extended to locally compact unimodular groups. In many cases, the L^2 -Betti numbers of a lattice are proportional to those of the ambient locally compact group, yielding new results for certain classes of lattices, including arithmetic lattices and Kac–Moody lattices.

© 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

RÉSUMÉ

Cet article présente quelques résultats de la thèse de l'auteur, « L^2 -Betti Numbers of Locally Compact Groups», dans laquelle la définition des nombres L^2 de Betti des groupes dénombrables a été généralisée au cas des groupes localement compacts unimodulaires. Dans de nombreux cas, les nombres L^2 de Betti d'un réseau quelconque dans un groupe localement compact sont proportionnels à ceux du groupe localement compact ambiant. On peut donc utiliser des théories puissantes concernant certaines classes de groupes localement compacts pour obtenir des calculs des nombres L^2 de Betti de leurs réseaux, en particulier les réseaux arithmétiques et les réseaux de Kac-Moody.

© 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

Recall that for a countable discrete group Γ , one can define [3,11] the L^2 -Betti numbers, $\beta_{(2)}^n(\Gamma)$, $n \ge 0$, as the (extended) von Neumann dimension of the group cohomology spaces:

 $\beta_{(2)}^n(\Gamma) = \dim_{L\Gamma} H^n(\Gamma, \ell^2 \Gamma).$

The dimension function $\dim_{L\Gamma}$ is Lück's extended von Neumann dimension [9,10], which with any module (in the purely algebraic sense) *E* over the group von Neumann algebra $L\Gamma$ of Γ associates an extended positive real number that agrees with the classical notion of von Neumann dimension when *E* is a Hilbert module, and enjoys in general many of the properties that one would expect a dimension function to have – see [11, Theorem 6.7].

1631-073X/\$ – see front matter © 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved. http://dx.doi.org/10.1016/j.crma.2013.05.013

E-mail address: henrik.petersen@epfl.ch.

¹ The work described here was completed while the author was supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92).

The theory of L^2 -Betti numbers provides in this way a set of powerful invariants that have a rich interplay with group theory, algebra, topology, geometry, and analysis. In [13], the framework is extended to include locally compact, 2nd countable, unimodular (lcsu) groups. Let *G* be an lcsu group with a fixed Haar measure λ . For any integer $n \ge 0$, the *n*th continuous cohomology $H^n(G, E)$ of *G* with coefficients in a locally convex continuous *G*-module *E*, where *E* is a locally convex topological vector space endowed with an action of *G* such that the map $G \times E \to E$ given by the action is continuous, is defined as the *n*th cohomology of the complex of inhomogeneous continuous cochains on *G*:

$$0 \to E \xrightarrow{d^0} C(G, E) \xrightarrow{d^1} C(G^2, E) \xrightarrow{d^2} \cdots$$

(See [1,7] for background on continuous cohomology.) Whenever *E* has a right-action of *LG* commuting with the leftaction of *G*, the cohomology spaces $H^n(G, E)$ are naturally right-*LG*-modules as well and, after extending the framework of Lück's von Neumann dimension to the case of semi-finite von Neumann algebras [13, Appendix B], we may define the L^2 -Betti numbers of *G* as [13, Chapter 3]:

$$\beta_{(2)}^n(G,\lambda) := \dim_{LG} H^n(G,L^2G).$$

Note that the dimension function in this case depends on the choice of scaling of the Haar measure, which is not necessarily canonical in this generality.

Remark 1. Observe that when *G* is totally disconnected, we may normalise the Haar measure such that some compact open subgroup has mass one. Thus the L^2 -Betti numbers are in this case canonical up to multiplication by strictly positive rationals.

When *G* is an lcsu group acting continuously and with compact stabilisers on a locally finite graph Δ , our definition agrees with Gaboriau's definition [6] of the first L^2 -Betti number of Δ . (See [13, Chapter 5].)

2. Lattices in lcsu groups

In his groundbreaking work [5] establishing the measure equivalence invariance of L^2 -Betti numbers, Gaboriau notes, in particular, that this implies the following special case: whenever Γ , Λ are lattices in an ambient locally compact group G, one has $\beta_{(2)}^n(\Gamma) = \frac{\text{covol}(\Gamma)}{\text{covol}(\Lambda)} \cdot \beta_{(2)}^n(\Lambda)$ for all $n \ge 0$. Since it was already well known, and straightforward to see, e.g. using the Shapiro lemma in group cohomology, that for a finite index inclusion $\Gamma \subseteq \Lambda$, one has in particular $\beta_{(2)}^n(\Gamma) = [\Lambda:\Gamma] \cdot \beta_{(2)}^n(\Lambda)$, it is natural to ask whether it holds that

(1)

$$\beta_{(2)}^n(\Gamma) = \operatorname{covol}_{\lambda}(\Gamma) \cdot \beta_{(2)}^n(G, \lambda)$$

whenever Γ is a lattice in an lcsu group *G*.

Theorem 2. (See [13, Theorems 4.8, 5.9].) Suppose that *G* is an lcsu group such that at least one of the following conditions holds:

- (i) G admits a cocompact lattice.
- (ii) G is totally disconnected.

Then Eq. (1) holds for any lattice Γ in G.

The proof in case (i) relies on Gaboriau's theorem [5, Théoreme 6.3]. To wit, let Γ_0 be a cocompact lattice in *G*. Then, by the Shapiro lemma in continuous cohomology, there is an isomorphism (of $L\Gamma_0$ -modules) $H^n(G, L^2G) \xrightarrow{\sim} H^n(\Gamma_0, \ell^2\Gamma_0)$ for any *n*. Then it is just a matter of comparing the *LG*-dimension of the left-hand side to the $L\Gamma_0$ -dimension, which is straightforward in this case. The result then follows for any lattice by Gaboriau's theorem, as explained above.

In general, the map between cohomology spaces above is not an isomorphism, and so one needs additional arguments to ensure that it is an isomorphism "up to dimension". In case (ii), the point is that one may trivialise cocycles in $H^n(G, L^2G)$ on compact open subgroups of G. This allows, in effect, an approximation of the cohomology by finite dimensional Hilbert modules (in the sense of *LG*-dimension), on which the isomorphism up to dimension follows by elementary spectral theory.

In fact, using ergodic-theoretic results, Eq. (1) can be proven for any inclusion of any closed, finite covolume unimodular subgroup [8, Theorem B].

3. Computation of higher L^2 -Betti numbers

Whereas the first L^2 -Betti number $\beta_{(2)}^1(\Gamma)$ has many well-known relations to the structure of the countable discrete group Γ , and many explicit computations and tools are available (we mention in particular [14]), much less seems to be known about the higher L^2 -Betti numbers.

Several new computations of (higher) L^2 -Betti numbers for countable discrete groups are obtained using the previous theorem and, for example, the analysis of the L^2 -cohomology of buildings in [4]. First, we consider algebraic groups over non-Archimedean local fields:

Theorem 3. (See [13, Theorem 5.30].) For all $n \ge 1$, and any non-Archimedean local field K such that the residue field has sufficiently large cardinality (depending on n), we have

$$\beta_{(2)}^n \left(Sp_{2n}(K) \right) > 0,$$

whence in particular $\beta_{(2)}^n(\Gamma) > 0$ for any lattice Γ in G.

When *K* is a function field $K = \mathbf{F}_q((t))$, it is known that $Sp_{2n}(K)$ does not admit any cocompact lattice (but does admit non-cocompact lattices, e.g. $Sp_{2n}(\mathbf{F}[t^{-1}])$), which is the interesting case.

Moreover, we also obtain non-vanishing results for Kac–Moody lattices. This follows by the results above and a Künneth formula for (totally disconnected) lcsu groups. (In fact, the Künneth formula holds in general for any product of lcsu groups, as is easily derived from the results in [13] and the structure theory.)

Theorem 4. (See [13, Theorem 6.9].) Let \mathbf{F}_q be a finite field of cardinality q, $G(\mathbf{F}_q)$ a complete Kac–Moody group and n the dimension of the building associated with $G(\mathbf{F}_q)$. Then, for q sufficiently large (depending on n):

$$\beta_{(2)}^{2n} \big(G(\mathbf{F}_q) \times G(\mathbf{F}_q) \big) > 0,$$

whence in particular $\beta_{(2)}^{2n}(\Gamma) > 0$ for any lattice Γ in $G(\mathbf{F}_q) \times G(\mathbf{F}_q)$.

Note that such lattices exist and are often simple and finitely generated, in some cases even finitely presented [2]. Thus these provide the first known examples of *simple* finitely generated (presented) countable groups with non-vanishing higher L^2 -Betti numbers.

4. A plaidoyer for totally disconnected groups

In the special case of totally disconnected lcsu groups, the existence of a local basis for the topology consisting of compact open subgroups in any such group, makes the theory of L^2 -Betti numbers very similar to the case of discrete groups. We refer to [13, Chapter 5] for details; it suffices to say here that the slogan is that one should expect to extend any result valid for discrete groups to totally disconnected lcsu groups, with the added book keeping of doing everything relative to a sequence of compact open subgroups decreasing to the identity. In particular, the proof of vanishing of L^2 -Betti numbers of amenable groups, due to Cheeger and Gromov [3] for countable groups, extends directly to totally disconnected groups [13, Theorem 5.37].

Next, every locally compact group *G* fits into a short exact sequence:

$$\mathbb{1} \to G_0 \to G \to G/G_0 \to \mathbb{1}$$

where G_0 is the connected component of the identity and G/G_0 is thus a totally disconnected group. If G is lcsu, then so are both of G_0 and G/G_0 . Using a well-known structural result (see e.g. [12, Theorem 11.3.4]), we can (virtually) replace the short exact sequence above with a direct product of a semisimple Lie group and a totally disconnected group, after passing to quotients modulo the amenable radical in G. Hence, modulo the amenable radical, the analysis is essentially reduced, at least in principle, to the case of totally disconnected groups. See [13, Theorem 7.12] for precise statements.

What remains is to handle lcsu groups with non-compact amenable radical. This is done in [8, Corollary E].

References

- Armand Borel, Nolan R. Wallach, Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups, Ann. Math. Stud., vol. 94, Princeton University Press, Princeton, NJ, 1980.
- [2] Pierre-Emmanuel Caprace, Bertrand Rémy, Simplicity and superrigidity of twin building lattices, Invent. Math. 176 (1) (2009) 169–221, http://dx.doi.org/ 10.1007/s00222-008-0162-6.
- [3] Jeff Cheeger, Mikhael Gromov, L_2 -cohomology and group cohomology, Topology 25 (2) (1986) 189–215, http://dx.doi.org/10.1016/0040-9383(86) 90039-X.
- [4] Jan Dymara, Tadeusz Januszkiewicz, Cohomology of buildings and their automorphism groups, Invent. Math. 150 (3) (2002) 579–627, http://dx.doi.org/ 10.1007/s00222-002-0242-y.
- [5] Damien Gaboriau, Invariants l² de relations d'équivalence et de groupes, Publ. Math. Inst. Hautes Études Sci. 95 (2002) 93–150, http://dx.doi.org/ 10.1007/s102400200002.
- [6] Damien Gaboriau, Invariant percolation and harmonic Dirichlet functions, Geom. Funct. Anal. 15 (5) (2005) 1004–1051, http://dx.doi.org/10.1007/ s00039-005-0539-2.
- [7] Alain Guichardet, Cohomologie des groupes topologiques et des algèbres de Lie, Textes Math. (Mathematical Texts), vol. 2, CEDIC, Paris, 1980.

- [8] David Kyed, Henrik Densing Petersen, Stefaan Vaes, L²-Betti numbers of locally compact groups and their cross section equivalence relations, preprint, arXiv:1302.6753.
- [9] Wolfgang Lück, Dimension theory of arbitrary modules over finite von Neumann algebras and L^2 -Betti numbers. I. Foundations, J. Reine Angew. Math. 495 (1998) 135–162.
- [10] Wolfgang Lück, Dimension theory of arbitrary modules over finite von Neumann algebras and L^2 -Betti numbers. II. Applications to Grothendieck groups, L^2 -Euler characteristics and Burnside groups, J. Reine Angew. Math. 496 (1998) 213–236.
- [11] Wolfgang Lück, L²-Invariants: Theory and Applications to Geometry and K-Theory, Ergeb. Math. Grenzgeb. (3) (Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics), vol. 44, Springer-Verlag, Berlin, 2002.
- [12] Nicolas Monod, Continuous Bounded Cohomology of Locally Compact Groups, Lect. Notes Math., vol. 1758, Springer-Verlag, Berlin, 2001.
- [13] Henrik Densing Petersen, L^2 -Betti numbers of locally compact groups, Thesis, University of Copenhagen, 2013, arXiv:1104.3294.
- [14] Jesse Peterson, Andreas Thom, Group cocycles and the ring of affiliated operators, Invent. Math. 185 (3) (2011) 561-592, http://dx.doi.org/10.1007/ s00222-011-0310-2.