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Differential Geometry

Isometric deformations of the $\mathcal{K}^{\frac{1}{4}}$-flow translators in $\mathbb{R}^3$ with helicoidal symmetry ☆

Déformations isométriques des translateurs du flot $\mathcal{K}^{1/4}$ dans $\mathbb{R}^3$ à symétrie hélicoïdale

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ABSTRACT

The height functions of $\mathcal{K}^{\frac{1}{4}}$-flow translators in the Euclidean space $\mathbb{R}^3$ solve the classical Monge–Ampère equation $f_{xx}f_{yy} - f_{xy}^2 = 1$. We explicitly and geometrically determine the moduli space of all helicoidal $\mathcal{K}^{\frac{1}{4}}$-flow translators, which are generated from planar curves by the action of helicoidal groups.

RÉSUMÉ

Les fonctions de hauteur des translateurs du flot $\mathcal{K}^{1/4}$ de $\mathbb{R}^3$ résolvent l'équation de Monge–Ampère classique $f_{xx}f_{yy} - f_{xy}^2 = 1$. Nous déterminons de manière géométrique explicite l'espace des modules de tous les translateurs à symétrie hélicoïdale du flot $\mathcal{K}^{1/4}$, qui sont engendré à partir de courbes planes par l'action de groupes hélicoïdaux.

1. Motivation and main results

1.1. Introduction

The recent decades saw intensive research devoted to the study of various geometric flows and soliton solutions. The classical curve-shortening flow admits fruitful generalizations with intriguing applications. One of Huisken's theorems guarantees that an analogue of the Gage–Hamilton's shrinking-curves theorem in the plane also holds for the mean curvature flow in higher dimensional Euclidean spaces.

Chow [6] studied the $\mathcal{K}^{\alpha}$-flow, which is the normal deformation by powers of the Gauss curvature. Given a smooth immersion $\mathcal{F}_0 : \Sigma \to \mathbb{R}^{n+1}$ of a strictly convex hypersurface Σ in Euclidean space $\mathbb{R}^{n+1}$, the solution of the initial value problem for the $\mathcal{K}^{\alpha}$-flow means a one-parameter family of smooth immersions $\{\mathcal{F}_t = \mathcal{F}(\cdot, t) : \Sigma \to \mathbb{R}^{n+1}\}_{t \in [0,T)}$ satisfying the geometric evolution:

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$$\begin{cases} \dfrac{\partial}{\partial t}\mathcal{F}(\mathbf{p},t) = -\mathcal{K}(\mathbf{p},t)^{\alpha}\nu(\mathbf{p},t), & (\mathbf{p},t)\in\Sigma\times[0,T),\\ \mathcal{F}(\mathbf{p},0) = \mathcal{F}_0(\mathbf{p}), \end{cases}$$

where $\alpha > 0$ is a constant, $\nu(\mathbf{p},t)$ denotes the outward-pointing unit normal of $\mathcal{F}(\mathbf{p},t)$, and the Gauss–Kronecker curvature functional $\mathcal{K}(\mathbf{p},t)$ is defined as the product of the principal curvatures. The Gauss curvature flow means the $\mathcal{K}^1$-flow which was investigated by Tso [23]. The case when $(n,\alpha) = (2,1)$ was originally introduced by Firey [11] in his study of a model of the wearing process of convex stones on a beach by water waves. In 1999, Andrews [3] established Firey's conjecture that convex surfaces evolving by the Gauss curvature flow become spherical.

For several motivations for the study of hypersurfaces moving by their Gauss curvature, we refer to [4, Section 1]. In particular, the $\mathcal{K}^{\frac{1}{n+2}}$-flow in Euclidean space $\mathbb{R}^{n+1}$ admits a deep and interesting geometric meaning in affine differential geometry. As indicated in [2, Section 1], the affine-invariant evolution of convex hypersurfaces in $\mathbb{R}^{n+1}$ under the so-called affine normal flow can be reformulated as the $\mathcal{K}^{\frac{1}{n+2}}$-flow modulo diffeomorphisms. It is worth to mention that the geometric meaning of the mysterious factor $\mathcal{K}^{\frac{1}{n+2}}$ in the classical affine differential geometry is also well-described in [9] and [5, Section 1] with details.

Urbas [24] investigated self-similar and translating solitons for the normal evolution by positive powers of the Gauss curvature. The simplest example of a translating soliton is Calabi's grim reaper $y = \ln(\cos x)$, which moves by downward translation under the $\mathcal{K}^1$-flow in the plane $\mathbb{R}^2$. In this paper, we say that a surface Σ in $\mathbb{R}^3$ is a $\mathcal{K}^{\frac{1}{4}}$*-translator* when we have the geometric condition: $\mathcal{K}_\Sigma = \cos^4(\theta_\Sigma)$. Here, the scalar function $\mathcal{K}_\Sigma$ denotes the Gaussian curvature, and the third component $\cos(\theta_\Sigma) = \mathbf{n}_\Sigma\cdot(0,0,1)$ of the unit normal $\mathbf{n}_\Sigma$ is called the angle function on Σ. When the initial surface in $\mathbb{R}^3$ is a $\mathcal{K}^{\frac{1}{4}}$-translator, it moves by vertical translation under the normal evolution of the $\mathcal{K}^{\frac{1}{4}}$-flow [24, Section 4].

The $\mathcal{K}^{\frac{1}{4}}$-translators in the Euclidean space $\mathbb{R}^3$ are of significant geometrical interest. The convex graph $z = f(x,y)$ becomes a $\mathcal{K}^{\frac{1}{4}}$-translator if and only if its height function f solves the classical Monge–Ampère equation:

$$f_{xx}f_{yy} - f_{xy}^2 = 1.$$

Jörgens' outstanding holomorphic resolution [16] says that, when $f_{xx}f_{yy} - f_{xy}^2 = 1$, the gradient graph (x,y,f_x,f_y) becomes a minimal surface in the Euclidean space $\mathbb{R}^4$. The Hessian one equation is a special case of special Lagrangian equations [14], *split* special Lagrangian equations [15,19,20], and affine mean curvature equations [2,5,22]. Furthermore, its solutions induce flat surfaces in hyperbolic space $\mathbb{H}^3$ [21].

1.2. Isometric deformations of helicoidal $\mathcal{K}^{\frac{1}{4}}$-translators

Theorem 1 *(Moduli space of $\mathcal{K}^{\frac{1}{4}}$-translators with rotational or helicoidal symmetry).*

(A) *Any helicoidal $\mathcal{K}^{\frac{1}{4}}$-translator Σ of pitch μ admits a one-parameter family of isometric helicoidal $\mathcal{K}^{\frac{1}{4}}$-translators Σ^h with pitch h such that $\Sigma = \Sigma^\mu$ and that Σ^0 is rotational.*

(B) *The cylinder over a circle in the xy-plane is a rotational $\mathcal{K}^{\frac{1}{4}}$-translator. Additionally, there exists a one-parameter family of $\mathcal{K}^{\frac{1}{4}}$-translators $\mathcal{H}_c$ invariant under the rotation with z-axis. The profile curve of rotational surface $\mathcal{H}_c$ is congruent to the graph $(U,0,\Lambda_c(U))$, where the one-parameter family of height functions $\Lambda_c(U)$ is explicitly given by*:

$$\Lambda_c(U) = \begin{cases} \frac{1}{2}[U\sqrt{U^2+\kappa^2} + \kappa^2\,\mathrm{arcsinh}(\frac{U}{\kappa})], & U > 0 \ (\textit{when } c = 1+\kappa^2,\ \kappa > 0),\\ \frac{1}{2}U^2, & U \geqslant 0 \ (\textit{when } c = 1),\\ \frac{1}{2}[U\sqrt{U^2-\kappa^2} - \kappa^2\,\mathrm{arccosh}(\frac{U}{\kappa})], & U > \kappa \ (\textit{when } c = 1-\kappa^2,\ \kappa > 0). \end{cases}$$

(C) *There exists a two-parameter family of helicoidal $\mathcal{K}^{\frac{1}{4}}$-translators $\mathcal{H}_c^h$ and the geometric coordinates (U,t) on $\mathcal{H}_c^h$ satisfying the following conditions.*

(C1) *The geometric meaning of parameter h is that the surface $\mathcal{H}_c^h$ is invariant under the helicoidal motion with pitch h. The surface $\mathcal{H}_c^h$ is invariant under the one-parameter subgroup $\{\mathbf{S}_T\}$ of the group of rigid motions of $\mathbb{R}^3 = \mathbb{C}\times\mathbb{R}$ given by:*

$$(\zeta,z)\in\mathbb{C}\times\mathbb{R}\mapsto\mathbf{S}_T(\zeta,z) = \left(e^{iT}\zeta, hT+z\right)\in\mathbb{C}\times\mathbb{R}.$$

(C2) *There exist the coordinates (U,t) on the helicoidal surface $\mathcal{H}_c^h$ such that its metric reads $I_{\mathcal{H}_c^h} = (U^2+c)\,\mathrm{d}U^2 + U^2\,\mathrm{d}t^2$.*

(C3) *The geometric meaning of parameter c is the property that the helicoidal surface $\mathcal{H}_c^h$ is isometric to the rotational surface $\mathcal{H}_c^0 = \mathcal{H}_c$.*

(C4) *The geometric meaning of coordinate U is the property that the function $\frac{1}{\sqrt{U^2+c}}$ coincides with the angle function on the surface $\mathcal{H}_c^h$ up to a sign.*

The statement (A) in Theorem 1 is inspired by the 1982 do Carmo–Dajczer theorem [8] that a surface of non-zero constant mean curvature is helicoidal if and only if it lies in the associate family [18] of a Delaunay's rotational surface [10,17] with the same constant mean curvature. In 1998, Haak [12] presented an alternative proof of the do Carmo–Dajczer theorem.

The *mean curvature flow* in $\mathbb{R}^3$ also admits the translating solitons with helicoidal symmetry. In 1994, Altschuler and Wu [1] showed the existence of the convex, rotational, entire graphical translator. In 2007, Clutterbuck, Schnürer and Schulze [7] constructed the bigraphical translator, which is also rotationally symmetric.

Open problem. Prove or disprove that Halldorsson's helicoidal translators [13] for the mean curvature flow admit the isometric deformation from rotational translators.

2. Proof of Theorem 1

We first need to revisit Bour's construction [8] with details to specify the behavior of the angle function on his isometric helicoidal surfaces.

Lemma 2 *(Angle function on Bour's helicoidal surfaces). Let Σ be a helicoidal surface with pitch vector $\mu\mathbf{k} = (0, 0, \mu)$ and the generating curve $\gamma = (\mathcal{R}, 0, \Lambda)$ in the xz-plane, which admits the parameterization $(u, \theta) \mapsto (\mathcal{R}\cos\theta, \mathcal{R}\sin\theta, \Lambda + \mu\theta)$, where u denotes a parameter of the generating curve γ. We then define the Bour coordinate transformation*:

$$(u, \theta) \mapsto (s, t) = (s, \theta + \Theta),$$

via the relations

$$\begin{cases} \mathrm{d}s^2 = \mathrm{d}\mathcal{R}^2 + \dfrac{\mathcal{R}^2}{\mathcal{R}^2 + \mu^2}\,\mathrm{d}\Lambda^2, \\ \mathrm{d}\Theta = \dfrac{\mu}{\mathcal{R}^2 + \mu^2}\,\mathrm{d}\Lambda, \end{cases}$$

and also introduce the Bour function U using the relation $U^2 = R^2 + \mu^2$.

(A) *The helicoidal surface Σ admits the reparametrization satisfying* (A1), (A2)*, and* (A3):

$$(s, t) \mapsto \mathbf{X}(s, t) = \big(\mathcal{R}\cos(t - \Theta), \mathcal{R}\sin(t - \Theta), \Lambda + \mu(t - \Theta)\big).$$

(A1) *Its first fundamental form reads $I_\Sigma = \mathrm{d}s^2 + U^2\,\mathrm{d}t^2$.*

(A2) *The parameters R, Λ, and Θ can be recovered from the Bour function U explicitly*:

$$\begin{cases} \mathcal{R}^2 = U^2 - \mu^2, \\ \mathrm{d}\Lambda^2 = \dfrac{U^2}{(U^2 - \mu^2)^2}\left(U^2\left(1 - \left(\dfrac{\mathrm{d}U}{\mathrm{d}s}\right)^2\right) - h^2\right)\mathrm{d}s^2, \\ \mathrm{d}\Theta = \dfrac{\mu}{U^2}\,\mathrm{d}\Lambda. \end{cases}$$

(A3) *The angle function n_3 defined as the third component $\mathbf{n} \cdot \mathbf{k}$ of the induced unit normal $\mathbf{n} = \frac{1}{\|\mathbf{X}_s \times \mathbf{X}_t\|}\mathbf{X}_s \times \mathbf{X}_t$ is also determined by the Bour function U.*

$$n_3^2 = \left(\frac{\mathrm{d}U}{\mathrm{d}s}\right)^2.$$

(B) *We construct a two-parameter family of helicoidal surfaces $\Sigma^{\lambda,h}$ of pitch h by the patch*:

$$\mathbf{X}^{\lambda,h}(s, t) = \left(\mathcal{R}^{\lambda,h}\cos\left(\frac{t}{\lambda} - \Theta^{\lambda,h}\right), \mathcal{R}^{\lambda,h}\sin\left(\frac{t}{\lambda} - \Theta^{\lambda,h}\right), \Lambda^{\lambda,h} + h\left(\frac{t}{\lambda} - \Theta^{\lambda,h}\right)\right),$$

where the geometric datum $(\mathcal{R}^{\lambda,h}, \Lambda^{\lambda,h}, \Theta^{\lambda,h})$ is explicitly determined by the pair (λ, h) of constants and the Bour function $U(s)$ arising from the reparametrization $\mathbf{X}(s, t)$ of Σ

$$\begin{cases} \left(\mathcal{R}^{\lambda,h}\right)^2 = \lambda^2 U^2 - h^2, \\ \left(\mathrm{d}\Lambda^{\lambda,h}\right)^2 = \dfrac{\lambda^2 U^2}{(\lambda^2 U^2 - h^2)^2}\left(\lambda^2 U^2\left(1 - \lambda^2\left(\dfrac{\mathrm{d}U}{\mathrm{d}s}\right)^2\right) - h^2\right)\mathrm{d}s^2, \\ \mathrm{d}\Theta^{\lambda,h} = \dfrac{h}{\lambda^2 U^2}\,\mathrm{d}\Lambda^{\lambda,h}. \end{cases} \tag{2.1}$$

Then, the helicoidal surface $\Sigma^{\lambda,h}$ is isometric to the initial surface Σ, and its angle function $n_3^{\lambda,h} = \mathbf{n}^{\lambda,h} \cdot \mathbf{k}$ is determined by the Bour function U of the initial surface Σ.

$$\left(n_3^{\lambda,h}\right)^2 = \lambda^2 \left(\frac{\mathrm{d}U}{\mathrm{d}s}\right)^2.$$

(C) *Furthermore, the helicoidal surface $\Sigma^{1,\mu}$ coincides with the initial surface Σ.*

Proof. (A) The definitions of the Bour coordinate (s,t) and the Bour function U yield:

$$\begin{aligned} I_\Sigma &= \left(\mathrm{d}R^2 + \mathrm{d}\Lambda^2\right) + 2\mu\,\mathrm{d}\Lambda\,\mathrm{d}\theta + \left(\mathcal{R}^2 + \mu^2\right)\mathrm{d}\theta^2 \\ &= \left(\mathrm{d}\mathcal{R}^2 + \frac{\mathcal{R}^2}{\mathcal{R}^2+\mu^2}\,\mathrm{d}\Lambda^2\right) + \left(\mathcal{R}^2+\mu^2\right)\left(\mathrm{d}\theta + \frac{\mu}{\mathcal{R}^2+\mu^2}\,\mathrm{d}\Lambda\right)^2 \\ &= \mathrm{d}s^2 + U^2\,\mathrm{d}t^2. \end{aligned}$$

Noticing that the definition $U^2 = R^2 + \mu^2$ implies $\mathrm{d}\mathcal{R}^2 = \frac{U^2}{U^2-\mu^2}\,\mathrm{d}U^2$, we can recover the function $\dot{\Lambda} = \frac{\mathrm{d}\Lambda}{\mathrm{d}s}$ from the Bour function $U(s)$ explicitly:

$$\mathrm{d}s^2 = \mathrm{d}\mathcal{R}^2 + \frac{\mathcal{R}^2}{\mathcal{R}^2+\mu^2}\,\mathrm{d}\Lambda^2 = \frac{U^2}{U^2-\mu^2}\,\mathrm{d}U^2 + \frac{U^2-\mu^2}{U^2}\,\mathrm{d}\Lambda^2,$$

and

$$\mathrm{d}\Lambda^2 = \frac{U^2}{U^2-\mu^2}\left(\mathrm{d}s^2 - \frac{U^2}{U^2-\mu^2}\,\mathrm{d}U^2\right) = \frac{U^2}{(U^2-\mu^2)^2}\left(U^2\left(1 - \left(\frac{\mathrm{d}U}{\mathrm{d}s}\right)^2\right) - h^2\right)\mathrm{d}s^2.$$

Adopting the symbol $\dot{} = \frac{\mathrm{d}}{\mathrm{d}s}$ again, we obtain:

$$\mathbf{X}_s \times \mathbf{X}_t = (\mu\dot{\mathcal{R}}\sin\theta - \mathcal{R}\dot{\Lambda}\cos\theta, -\mu\dot{\mathcal{R}}\cos\theta - \mathcal{R}\dot{\Lambda}\sin\theta, \mathcal{R}\dot{\mathcal{R}}).$$

After setting $I_\Sigma := E\,\mathrm{d}s^2 + 2F\,\mathrm{d}s\,\mathrm{d}t + G\,\mathrm{d}t^2 = \mathrm{d}s^2 + U^2\,\mathrm{d}t^2$, we immediately see that: $\|\mathbf{X}_s \times \mathbf{X}_t\|^2 = EG - F^2 = U^2$. It thus follows that:

$$n_3^2 = \frac{(\mathcal{R}\dot{\mathcal{R}})^2}{U^2} = \dot{U}^2 = \left(\frac{\mathrm{d}U}{\mathrm{d}s}\right)^2.$$

(B) We first show that the surface $\Sigma^{\lambda,h}$ is isometric to the initial surface Σ. Let us write: $I_{\Sigma^{\lambda,h}} = E^{\lambda,h}\,\mathrm{d}s^2 + 2F^{\lambda,h}\,\mathrm{d}s\,\mathrm{d}t + G^{\lambda,h}\,\mathrm{d}t^2$. Adopting the symbol $\dot{} = \frac{\mathrm{d}}{\mathrm{d}s}$ and using (2.1), we have:

$$\begin{aligned} E^{\lambda,h} &= \left(\dot{\mathcal{R}}^{\lambda,h}\right)^2 + \mathcal{R}^2\left(\dot{\Theta}^{\lambda,h}\right)^2 + \left(\dot{\Lambda}^{\lambda,h} - h\dot{\Theta}^{\lambda,h}\right)^2 = \left(\dot{\mathcal{R}}^{\lambda,h}\right)^2 + \frac{\lambda^2U^2-h^2}{\lambda^2U^2}\left(\dot{\Theta}^{\lambda,h}\right)^2 \\ &= \frac{\lambda^2U^2\dot{U}^2}{\lambda^2U^2-h^2} + \frac{\lambda^2U^2-h^2}{\lambda^2U^2} \cdot \frac{\lambda^2U^2[\lambda^2U^2(1-\lambda^2\dot{U}^2)-h^2]}{\left(\lambda^2U^2-h^2\right)^2} = 1. \end{aligned}$$

We also deduce:

$$F^{\lambda,h} = -\frac{1}{\lambda}\left[\left(\left(\mathcal{R}^{\lambda,h}\right)^2 + h^2\right)\dot{\Theta} - h\dot{\Lambda}\right] = -\frac{1}{\lambda}\left[\lambda^2U^2\dot{\Theta} - h\dot{\Lambda}\right] = 0,$$

and

$$G^{\lambda,h} = \frac{1}{\lambda^2}\left[\left(\mathcal{R}^{\lambda,h}\right)^2 + h^2\right] = U^2.$$

Combining these, we meet $I_{\Sigma^{\lambda,h}} = E^{\lambda,h}\,\mathrm{d}s^2 + 2F^{\lambda,h}\,\mathrm{d}s\,\mathrm{d}t + G^{\lambda,h}\,\mathrm{d}t^2 = \mathrm{d}s^2 + U^2\,\mathrm{d}t^2 = I_\Sigma$. Now, it remains to determine the angle function of the surface $\Sigma^{\lambda,h}$. Adopting the new variable $\theta = \frac{t}{\lambda} - \Theta^{\lambda,h}$ for simplicity, we write:

$$\mathbf{X}_s^{\lambda,h} \times \mathbf{X}_t^{\lambda,h} = \frac{1}{\lambda}\left(h\dot{\mathcal{R}}^{\lambda,h}\sin\theta - \mathcal{R}^{\lambda,h}\dot{\Lambda}\cos\theta, -h\dot{\mathcal{R}}^{\lambda,h}\cos\theta - \mathcal{R}^{\lambda,h}\dot{\Lambda}\sin\theta, \mathcal{R}^{\lambda,h}\dot{\mathcal{R}}^{\lambda,h}\right).$$

Taking account into this and the equality $\|\mathbf{X}_s^{\lambda,h} \times \mathbf{X}_t^{\lambda,h}\|^2 = E^{\lambda,h}G^{\lambda,h} - (F^{\lambda,h})^2 = U^2$, we meet

$$\left(n_3^{\lambda,h}\right)^2 = \left(\mathbf{n}^{\lambda,h} \cdot \mathbf{k}\right)^2 = \frac{1}{U^2} \cdot \frac{\left(\mathcal{R}^{\lambda,h}\right)^2\left(\dot{\mathcal{R}}^{\lambda,h}\right)^2}{\lambda^2} = \lambda^2\dot{U}^2 = \lambda^2\left(\frac{\mathrm{d}U}{\mathrm{d}s}\right)^2.$$

(C) The datum $(\mathcal{R}^{1,\mu}, \Lambda^{1,\mu}, \Theta^{1,\mu})$ of $\Sigma^{1,\mu}$ coincides with the datum $(\mathcal{R}, \Lambda, \Theta)$ of Σ. □

We briefly sketch the geometric ingredients in our construction in Theorem 1. For given a helicoidal $\mathcal{K}^{\frac{1}{4}}$-translator, we prove that there exists a sub-family chosen from the two-parameter family of Bour's isometric helicoidal surfaces, so that each member of this sub-family is a $\mathcal{K}^{\frac{1}{4}}$-translator and that one member is rotationally symmetric.

Our one-parameter family of $\mathcal{K}^{\frac{1}{4}}$-translators admits the parametrizations by so-called the Bour coordinate (s,t) and the Bour function $U = U(s)$. The trick to obtain the explicit construction in (C) of Theorem 1 is to perform the coordinate transformation $s \mapsto U$ to have the geometric coordinate (U,t) on our one-parameter family of $\mathcal{K}^{\frac{1}{4}}$-translators.

Lemma 3 *(Existence of helicoidal $\mathcal{K}^{\frac{1}{4}}$-translators of pitch h). Let h be a given constant. Then, any non-cylindrical helicoidal $\mathcal{K}^{\frac{1}{4}}$-translator with pitch h admits the parameterization*:

$$(U,t) \mapsto \big(\mathcal{R}(U)\cos\big(t-\Theta(U)\big),\ \mathcal{R}(U)\sin\big(t-\Theta(U)\big),\ \Lambda(U)+h\big(t-\Theta(U)\big)\big),$$

where the geometric datum $(\mathcal{R}(U), \Lambda(U), \Theta(U))$ can be obtained from the relation:

$$\begin{cases} \mathcal{R}(U)^2 = U^2 - h^2, \\ \left(\dfrac{\mathrm{d}\Lambda}{\mathrm{d}U}\right)^2 = \dfrac{U^2}{(U^2-h^2)^2}\big[U^4 + \big(c-1-h^2\big)U^2 - h^2 c\big], \\ \left(\dfrac{\mathrm{d}\Theta}{\mathrm{d}U}\right)^2 = \dfrac{h^2}{U^2(U^2-h^2)^2}\big[U^4 + \big(c-1-h^2\big)U^2 - h^2 c\big], \end{cases} \tag{2.2}$$

where $c \in \mathbb{R}$ is a constant.

Proof. Taking $\lambda = 1$ in Lemma 2, we construct a helicoidal surface Σ with pitch h:

$$(s,t) \mapsto \mathbf{X}^{1,h}(s,t) = \big(\mathcal{R}\cos(t-\Theta), \mathcal{R}\sin(t-\Theta), \Lambda + h(t-\Theta)\big),$$

where the geometric datum $(\mathcal{R}, \Lambda, \Theta) = (\mathcal{R}(s), \Lambda(s), \Theta(s))$ is given by the relation:

$$\begin{cases} \mathcal{R}^2 = U^2 - h^2, \\ (\mathrm{d}\Lambda)^2 = \dfrac{U^2}{(U^2-h^2)^2}\left(U^2\left(1-\left(\dfrac{\mathrm{d}U}{\mathrm{d}s}\right)^2\right) - h^2\right)\mathrm{d}s^2, \\ \mathrm{d}\Theta = \dfrac{h}{U^2}\,\mathrm{d}\Lambda. \end{cases} \tag{2.3}$$

The key point is to take the Bour function U as the new parameter on our helicoidal surface Σ. According to Lemma 2 again, we see that the induced metric on Σ reads $I_\Sigma = \mathrm{d}s^2 + U^2\,\mathrm{d}t^2$, that its Gaussian curvature K is equal to $K = -\frac{1}{U}\frac{\mathrm{d}^2U}{\mathrm{d}s^2}$, and that its angle function reads $n_3^2 = (\frac{\mathrm{d}U}{\mathrm{d}s})^2$. Thus, the condition that the helicoidal surface Σ becomes a $\mathcal{K}^{\frac{1}{4}}$-translator implies that $K = n_3^4$, which means the ordinary differential equation:

$$-\frac{1}{U}\frac{\mathrm{d}^2U}{\mathrm{d}s^2} = \left(\frac{\mathrm{d}U}{\mathrm{d}s}\right)^4.$$

In the case when $\frac{\mathrm{d}U}{\mathrm{d}s}$ vanishes locally, our surface Σ becomes the cylinder over a circle in the xy-plane. When $\frac{\mathrm{d}U}{\mathrm{d}s}$ does not vanish, we are able to make a coordinate transformation $s \mapsto U$ and can rewrite the above ODE as: $0 = \frac{\mathrm{d}}{\mathrm{d}s}(1/(\frac{\mathrm{d}U}{\mathrm{d}s})^2 - U^2)$. Hence its first integral is explicitly given by, for some constant $c \in \mathbb{R}$, $\mathrm{d}s^2 = (U^2 + c)\,\mathrm{d}U^2$. We now can employ this to perform the coordinate transformation $(s,t) \mapsto (U,t)$ on Σ. Rewriting (2.3) in terms of the new variable U gives indeed the relation in (2.2). □

Proof of Theorem 1. We first prove (B). Taking $h = 0$ in Lemma 3, we see that any rotational $\mathcal{K}^{\frac{1}{4}}$-translator admits the patch:

$$(U,t) \mapsto \big(\mathcal{R}(U)\cos\big(t-\Theta(U)\big),\ \mathcal{R}(U)\sin\big(t-\Theta(U)\big),\ \Lambda(U)+h\big(t-\Theta(U)\big)\big),$$

where the geometric datum $(\mathcal{R}(U), \Lambda(U), \Theta(U))$ satisfies the relation:

$$\big(\mathcal{R}(U)\big)^2 = U^2, \qquad \left(\frac{\mathrm{d}\Lambda}{\mathrm{d}U}\right)^2 = U^2 + (c-1), \qquad \left(\frac{\mathrm{d}\Theta}{\mathrm{d}U}\right)^2 = 0$$

for some constant $c \in \mathbb{R}$. The condition that the helicoidal surface Σ becomes a $\mathcal{K}^{\frac{1}{4}}$-translator implies the ordinary differential equation:

$$-\frac{1}{U}\frac{\mathrm{d}^2U}{\mathrm{d}s^2} = \left(\frac{\mathrm{d}U}{\mathrm{d}s}\right)^4.$$

When $\frac{\mathrm{d}U}{\mathrm{d}s}$ vanishes locally, our surface Σ becomes the cylinder over a circle in the xy-plane. In the case when $\frac{\mathrm{d}U}{\mathrm{d}s}$ does not vanish, we can introduce a coordinate transformation $s \mapsto U$. Since $\frac{\mathrm{d}\Theta}{\mathrm{d}U}$ vanishes, without loss of generality, after a translation of the coordinate t, we may take $\Theta = 0$ in the above patch as follows:

$$(U, t) \mapsto \big(U \cos t, U \sin t, \Lambda(U)\big).$$

As in the proof of Lemma 3, $\Lambda(U)$ solves the ordinary differential equation: $\frac{\mathrm{d}\Lambda}{\mathrm{d}U} = \pm\sqrt{U^2 + (c-1)}$. Considering the sign of the constant $c - 1$, we meet the explicit solution $\Lambda_c(U) = \Lambda(U)$ (up to the sign) as follows:

$$\Lambda(U) = \begin{cases} \frac{1}{2}[U\sqrt{U^2+\kappa^2} + \kappa^2 \operatorname{arcsinh}(\frac{U}{\kappa})] & (\text{when } c = 1+\kappa^2,\ \kappa > 0), \\ \frac{1}{2}U^2 & (\text{when } c = 1), \\ \frac{1}{2}[U\sqrt{U^2-\kappa^2} - \kappa^2 \operatorname{arccosh}(\frac{U}{\kappa})] & (\text{when } c = 1-\kappa^2,\ \kappa > 0). \end{cases}$$

We next prove (A). Using Lemma 2, we see that, for a given helicoidal $\mathcal{K}^{\frac{1}{4}}$-translator Σ, we are able to introduce the Bour coordinate (s,t) and the Bour function $U(s)$ on the surface Σ so that $I_\Sigma = \mathrm{d}s^2 + U(s)^2\,\mathrm{d}t^2$. The condition that Σ is a $\mathcal{K}^{\frac{1}{4}}$-translator says:

$$-\frac{1}{U}\frac{\mathrm{d}^2U}{\mathrm{d}s^2} = \left(\frac{\mathrm{d}U}{\mathrm{d}s}\right)^4, \tag{2.4}$$

just as we saw in the proof of Lemma 3. Next, by Lemma 2 again, we can associate a one-parameter family of isometric helicoidal surfaces Σ^h satisfying three conditions: $\Sigma = \Sigma^\mu$, $I_{\Sigma^h} = I_\Sigma$, and the angle function on Σ^h coincides with the one on Σ. Hence, as we saw in the proof of Lemma 3, the above ordinary differential equation in (2.4) guarantees that any helicoidal surface Σ^h becomes indeed a $\mathcal{K}^{\frac{1}{4}}$-translator.

It now remains to show (C). The statement (C1) is obvious by the construction in Lemma 3. Next, the equality $\mathrm{d}s^2 = (U^2 + c)\,\mathrm{d}U^2$ proved in Lemma 3 implies that the induced metric of the helicoidal surface constructed in Lemma 3 reads: $\mathrm{d}s^2 + U^2\,\mathrm{d}t^2 = (U^2+c)\,\mathrm{d}U^2 + U^2\,\mathrm{d}t^2$ (which implies (C2) and (C3)), and that the angle function is given by, up to a sign:

$$\frac{\mathrm{d}U}{\mathrm{d}s} = \frac{1}{\frac{\mathrm{d}s}{\mathrm{d}U}} = \frac{1}{\sqrt{U^2+c}},$$

which is (C4). This completes the proof of our description of the moduli space of helicoidal $\mathcal{K}^{\frac{1}{4}}$-translators in Theorem 1. □

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