C. R. Acad. Sci. Paris, Ser. I 351 (2013) 451–456

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial Differential Equations

On the existence of weak solutions to a model problem for the unsteady turbulent pipe-flow

Sur l'existence des solutions faibles pour un modèle d'écoulement non stationnaire turbulent dans une conduite

Joachim Naumann

Department of Mathematics, Humboldt University Berlin, Unter den Linden 6, 10099 Berlin, Germany

ARTICLE INFO

Article history:
Received 28 February 2013
Accepted 27 June 2013
Available online 23 July 2013

Presented by Philippe G. Ciarlet

ABSTRACT

We consider a coupled system of PDEs for the scalar functions u and k in a cylinder $\Omega\times\,]0,T[$ ($\Omega\subset\mathbb{R}^2$ bounded domain, $0<T<+\infty$). This system represents a simplified version of Prandtl's (1945) model of turbulence in the case of an unsteady motion of a fluid through a pipe with cross-section Ω (u = one-dimensional velocity, k = turbulent kinetic energy). We prove the existence of weak solutions to the problem under consideration with homogeneous Dirichlet conditions on u and homogeneous Neumann conditions on k along $\partial\Omega\times\,]0,T[$, and initial conditions on u and k in $\Omega\times\{0\}$.

RÉSUMÉ

On considère un système couplé d'équations aux dérivées partielles pour des fonctions scalaires u et k dans un cylindre $\Omega\times\,]0,T[$ ($\Omega\subset\mathbb{R}^2$ domaine borné, $0<T<+\infty$). Ce système représente une version simplifiée du modèle de turbulence de Prandtl (1945) dans le cas de l'écoulement non stationnaire d'un liquide dans une conduite de section Ω (u = vitesse à une dimension, k = énergie cinétique de la turbulence). Nous démontrons l'existence de solutions faibles pour le système envisagé avec des conditions homogènes de Dirichlet pour u et des conditions de Neumann pour k sur $\partial\Omega\times\,]0,T[$, et des conditions initiales pour des fonctions u et k dans $\Omega\times\{0\}$.

Version française abrégée

Dans un cylindre $Q=\Omega\times\,]0,T[$ ($\Omega\subset\mathbb{R}^2$ domaine borné de Lipschitz, $0<T<+\infty$), on étudie le système :

$$\frac{\partial u}{\partial t}-\operatorname{div}(\sqrt{k}\nabla u)=f,\qquad \frac{\partial k}{\partial t}-\operatorname{div}\big((\mu+\sqrt{k})\nabla k\big)=\sqrt{k}|\nabla u|^2-k\sqrt{k}, \tag{1}$$

où la fonction $f=f(t)$ (= gradient de la pression de l'écoulement) est donnée dans $]0,T[$, et $\mu=\text{const}>0$. Ce système est une version simplifiée du modèle de turbulence de Prandtl (1945) [18] pour l'écoulement d'un liquide dans une conduite de section Ω. La fonction $u=u(x,t)$ représente la vitesse à une dimension et $k=k(x,t)$ dénote l'énergie cinétique de la

E-mail address: jnaumann@math.hu-berlin.de.

1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2013.06.011

turbulence ($x=(x_1,x_2)\in\Omega$, $t\in\,]0,T[$; voir [5,16,17,20] pour les notions fondamentales). On considère les conditions aux limites et les conditions initales suivantes :

$$u=\frac{\partial k}{\partial \mathbf{n}}=0 \quad \text{sur } \partial\Omega\times\,]0,T[, \quad u=u_0, \quad k=k_0 \quad \text{dans } \Omega\times\{0\} \tag{2}$$

($\mathbf{n}$ = normale éxterieure).

L'objet de cette Note est la continuation des études qui ont commencé avec [14,15] pour les cas non stationnaires. Pour simplifier les raisonnements, nous nous limitons au cas où $f=0$. On démontre alors l'existence de solutions faibles pour le problème (1), (2), de sorte que $k\geqslant 0$ p.p. dans Q et $k>0$ p.p. dans $Q^*\subset Q$, où mes $Q^*>0$. La démonstration se fait en trois étapes :

1. On prouve l'existence d'une solution faible $(u_\varepsilon,k_\varepsilon)$ ($k_\varepsilon\geqslant 0$ p.p. dans Q) au système régularisé

$$\frac{\partial u}{\partial t}-\operatorname{div}\big((\varepsilon+[k]_\varepsilon)^{1/2}\nabla u\big)=0, \qquad \frac{\partial k}{\partial t}-\operatorname{div}\big((\mu+[k]_\varepsilon^{1/2})\nabla k\big)=(\varepsilon+[k]_\varepsilon)^{1/2}|\nabla u|^2-k^{3/2}$$

   sous les conditions (2), ($[\xi]_\varepsilon:=\min\{\frac{1}{\varepsilon},\xi\}$, $0\leqslant\xi<+\infty$, $0<\varepsilon<+\infty$).
2. On établit des estimations a priori pour $(u_\varepsilon,k_\varepsilon)$.
3. On effectue le passage à la limite pour $(u_\varepsilon,k_\varepsilon)$ lorsque $\varepsilon\to 0$. Pour cette étape, l'outil principal est l'équation d'énergie locale suivante pour la solution faible (u,k) de (1), (2) :

$$\frac{1}{2}\int_\Omega u^2(x,t)\zeta(x)\,\mathrm{d}x+\int_0^t\int_\Omega k^{1/2}\big(|\nabla u|^2\zeta+u\nabla u\cdot\nabla\zeta\big)=\frac{1}{2}\int_\Omega u_0^2\zeta \quad \forall t\in[0,T],\ \forall\zeta\in C_c^1(\Omega)$$

   qui a un intérêt indépendant (voir aussi (23)).

1. Introduction

Let $\Omega\subset\mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial\Omega$, let $0<T<+\infty$ and put $Q=\Omega\times\,]0,T[$. We consider the following system of PDEs:

$$\frac{\partial u}{\partial t}-\operatorname{div}(\sqrt{k}\nabla u)=f, \qquad \frac{\partial k}{\partial t}-\operatorname{div}\big((\mu+\sqrt{k})\nabla k\big)=\sqrt{k}|\nabla u|^2-k\sqrt{k} \quad \text{in } Q. \tag{1}$$

This system represents a simplified version of Prandtl's 1945 one-equation model of turbulence for the unsteady motion of an incompressible fluid through the "pipe" $\Omega\times\,]a,b[$ $(-\infty<a<b<+\infty)$. The simplification we made in [18, Eq. (I) on p. 11] consists in assuming that the mixing length l is constant, say 1 (cf. [16,17,20] for details concerning the concept of Prandtl's mixing length l; cf. also [15]). Finally, $f(t)$ denotes the given pressure gradient, i.e. $f(t)=-\frac{\partial p}{\partial x_3}(x_3,t)$, where $p(x_3,t)=(a-x_3)f(t)$, $x_3\in\,]a,b[$, $f\geqslant 0$ (cf. [1, pp. 179–180], [5, pp. 54–56], [15]). Detailed discussions of turbulence modeling can be found in [5,16,17,20].

We complete (1) by the following boundary and initial conditions:

$$u=\frac{\partial k}{\partial \mathbf{n}}=0 \quad \text{on } \partial\Omega\times\,]0,T[, \quad u=u_0, \quad k=k_0 \quad \text{in } \Omega\times\{0\}, \tag{2}$$

where $\mathbf{n}$ denotes the exterior unit normal to $\partial\Omega$, and u_0 and k_0 denote given functions in Ω.

In [10, pp. 203–204], the author considers a system of PDEs for two scalar functions, which is more complex than (1), but does not include a degenerate PDE like the u-equation in (1). In [4,9], the authors prove the existence of weak solutions to a general class of turbulent-viscosity models in three dimensions of space, with eddies of the type $\nu_0+\nu(k)$ ($\nu_0=\text{const}>0$, $0\leqslant\nu(k)\leqslant c_0(1+k^\alpha)$ for all $k\in[0,+\infty[$).

The aim of the present Note is to continue the discussion in [14,15]. We state an existence theorem for weak solutions to (1), (2) and sketch its proof. For the sake of simplicity of the discussion, in what follows we suppose that $f=0$.

2. Statement of the main result

Let X denote a real normed vector space with norm $|\cdot|_X$ and let X^* be its dual. By $\langle x^*,x\rangle_{X^*,X}$ we denote the dual pairing of $x^*\in X^*$ and $x\in X$. The symbol $C_w([0,T];X)$ stands for the vector space of all mappings $u:[0,T]\to X$ such that, for every $x^*\in X^*$, the function $t\mapsto\langle x^*,u(t)\rangle_{X^*,X}$ is continuous on $[0,T]$. Next, by $L^p(0,T;X)$ $(1\leqslant p\leqslant+\infty)$ we denote the vector space of all equivalence classes of measurable mappings $u:[0,T]\to X$ such that the function $t\mapsto|u(t)|_X$ is in $L^p(0,T)$ (cf. [2, Chap. III, §3, Chap. IV, §3], [3, Appendice], [6] for the basics of $L^p(0,T;X)$).

Let $W^{1,p}(\Omega)$ $(1\leqslant p\leqslant+\infty)$ denote the usual Sobolev space and let:

$$W_0^{1,p}(\Omega):=\big\{u\in W^{1,p}(\Omega);\ u=0 \text{ a.e. on } \partial\Omega\big\}, \quad W^{-1,p'}(\Omega):=\text{ dual of } W_0^{1,p}(\Omega) \quad \left(1<p<+\infty,\ p'=\frac{p}{p-1}\right).$$

The main result of our paper is the following:

Theorem. *Let $u_0 \in L^\infty(\Omega)$ and let $k_0 \in L^1(\Omega)$, $k_0 \geqslant 0$ a.e. in Ω. Then there exists a pair (u,k) such that*:

$$\left.\begin{aligned} &\min\Big\{0, \operatorname*{ess\,inf}_{\Omega} u_0\Big\} \leqslant u \leqslant \max\Big\{0, \operatorname*{ess\,sup}_{\Omega} u_0\Big\}, \quad k \geqslant 0 \text{ a.e. in } Q, \\ &u \in C_w\big([0,T]; L^2\Omega\big) \cap L^2\big(0,T; W_0^{1,2}(\Omega)\big), \quad u' \in L^2\big(0,T; W^{-1,4/3}(\Omega)\big), \end{aligned}\right\} \tag{3}$$

$$k \in L^\infty\big(0,T; L^1(\Omega)\big) \cap \Bigg(\bigcap_{1\leqslant q<\frac{16}{7}} L^q(Q)\Bigg), \quad \nabla k \in \bigcap_{1\leqslant r<\frac{4}{3}} \big[L^r(Q)\big]^2, \quad k' \in \bigcap_{8<s<+\infty} L^1\big(0,T; W^{-1,s'}(\Omega)\big), \tag{4}$$

$$\int_Q k^{1/2}|\nabla u|^2 < +\infty, \qquad \mu\int_Q \frac{|\nabla k|^2}{(1+k)^{1+\delta}} \leqslant C_1 \quad \forall\delta \in \,]0,1[\,, \qquad \int_Q \big|\nabla k^{3/2}\big|^{\tilde{s}} \leqslant C_2 \quad \forall \tilde{s} \in \Big[1, \frac{8}{7}\Big[\,, \tag{5}$$

where $C_1 = C_1(\delta) \to +\infty$ as $\delta \to 0$ or $\delta \to 1$, respectively, and $C_2 = C_2(\tilde{s}) \to +\infty$ as $\tilde{s} \to \frac{8}{7}$,

$$\int_0^T \big\langle u'(t), v(t)\big\rangle_{W^{-1,2}, W_0^{1,2}} \,dt + \int_Q k^{1/2}\nabla u\cdot\nabla v = 0 \quad \forall v \in L^2\big(0,T; W_0^{1,4}(\Omega)\big), \tag{6}$$

$$\left.\begin{aligned} &\text{for every } s \in \,]8,+\infty[\,, \ \int_0^T \big\langle k'(t), w(t)\big\rangle_{W^{-1,s'}, W_0^{1,s}} \,dt + \int_Q \Big(\mu\nabla k + \frac{2}{3}\nabla k^{3/2}\Big)\cdot\nabla w \\ &\qquad = \int_Q \big(k^{1/2}|\nabla u|^2 - k^{3/2}\big) w \quad \forall w \in L^\infty\big(0,T; W_0^{1,s}(\Omega)\big), \end{aligned}\right\} \tag{7}$$

and $u(0) = u_0$ a.e. in Ω, $k(0) = k_0$ in $(W^{1,s}(\Omega))^*$ for any $s \in \,]8,+\infty[$. In addition, (u,k) satisfies:

$$\frac{1}{2}\int_\Omega u^2(x,t)z(x)\,dx + \int_0^t\int_\Omega k^{1/2}\big(|\nabla u|^2 z + u\nabla u\cdot\nabla z\big) = \frac{1}{2}\int_\Omega u_0^2 z \quad \forall t \in [0,T], \ \forall z \in W_0^{1,4}(\Omega), \tag{8}$$

$$2\int_\Omega k_0^{1/2} + \int_0^t\int_\Omega |\nabla u|^2 \leqslant 2\int_\Omega k^{1/2}(x,t)\,dx + \int_0^t\int_\Omega k \quad \text{for a.e. } t \in [0,T], \tag{9}$$

$$\int_\Omega \Big(\frac{1}{2}u^2(x,t) + k(x,t)\Big)\,dx + \int_0^t\int_\Omega k^{3/2} \leqslant \int_\Omega \Big(\frac{1}{2}u_0^2 + k_0\Big)\,dx \quad \text{for a.e. } t \in [0,T]. \tag{10}$$

Remarks.

1. The derivatives u' in (3) and k' in (4) have to be understood in the sense of distributions from $]0,T[$ into the vector spaces $W^{-1,4/3}(\Omega)$ and $(W^{1,s}(\Omega))^*$, respectively (cf. [3, Appendix], [6]).
   We notice that (6) represents a weak formulation of the u-equation in (1) with boundary condition $u = 0$ on $\partial\Omega \times \,]0,T[$ (cf. [8] for a different weak formulation of the steady case of (1), (2) with boundary conditions $u = k = 0$ on $\partial\Omega$). Next, for the weak formulation of the k-equation in (1) with boundary condition $\frac{\partial k}{\partial \mathbf{n}} = 0$ on $\partial\Omega \times \,]0,T[$, it is natural to use test functions that are not subjected to boundary conditions. In the present paper, however, the use of the test function $w \in L^\infty(0,T; W_0^{1,s}(\Omega))$ in (7) is dictated by the local energy equality (8) (see part 4 of the sketch of proof).
2. Let be (u,k) as in the theorem. Then there exists $0 < t_0 \leqslant T$ such that $\|u_0\|_{L^2(\Omega)} \leqslant \frac{c_0}{t_0^2}\int_0^{t_0}(\int_0^t\int_\Omega |\nabla u|^2)$ ($c_0 = \text{const} > 0$). We may assume that (9) holds for t_0. Hence, for any $k_0 \in L^1(\Omega)$,
   $$\|u_0\|_{L^2(\Omega)} \leqslant \frac{c_0}{t_0^2}\max\{2, t_0\}\int_0^{t_0}\int_\Omega k^{1/2}\big(1+k^{1/2}\big).$$
   Thus, even in case of $\|k_0\|_{L^1(\Omega)} = 0$, from $\|u_0\|_{L^2(\Omega)} > 0$ it follows that there exists a measurable set $Q^* \subset \Omega \times \,]0,t_0[$ such that $\operatorname{mes} Q^* > 0$ and $k > 0$ a.e. in Q^*.
3. We conjecture that there exists a weak solution (u,k) to (1), (2) for which equality holds in (10) (cf. [13]).

3. Sketch of proof

For $0 \leqslant \xi < +\infty$ and $0 < \varepsilon < +\infty$, define $[\xi]_\varepsilon := \min\{\frac{1}{\varepsilon}, \xi\}$.

1° *Existence of approximate solutions.* Without loss of generality, we may assume $u_0 \in W_0^{1,2}(\Omega)$ and $k_0 \in W^{1,2}(\Omega)$, $k_0 \geqslant 0$ a.e. in Ω.

For every $\varepsilon > 0$ there exists a pair $(u_\varepsilon, k_\varepsilon) \in L^4(0,T;W_0^{1,4}(\Omega)) \times L^2(0,T;W^{1,2}(\Omega))$ such that:

$$\min\Big\{0, \operatorname*{ess\,inf}_{\Omega} u_0\Big\} \leqslant u_\varepsilon \leqslant \max\Big\{0, \operatorname*{ess\,sup}_{\Omega} u_0\Big\} \quad \text{a.e. in } Q, \qquad k_\varepsilon \geqslant 0 \quad \text{a.e. in } Q,$$

$$u_\varepsilon, k_\varepsilon \in C\big([0,T];L^2(\Omega)\big), \quad u'_\varepsilon \in L^2\big(0,T;W^{-1,2}(\Omega)\big), \quad k'_\varepsilon \in L^2\big(0,T;\big(W^{1,2}(\Omega)\big)^*\big),$$

$$\langle u'_\varepsilon(t), v\rangle_{W^{-1,2}W_0^{1,2}} + \int_\Omega \big(\big(\varepsilon + [k_\varepsilon]_\varepsilon\big)^{1/2} + \varepsilon|\nabla u_\varepsilon|^2\big)\nabla u_\varepsilon \cdot \nabla v = 0 \quad \text{for a.e. } t \in [0,T],\ \forall v \in W_0^{1,4}(\Omega), \tag{11}$$

$$\left.\begin{aligned} &\langle k'_\varepsilon(t), z\rangle_{(W^{1,2})^*, W^{1,2}} + \int_\Omega \big(\mu + [k_\varepsilon]_\varepsilon^{1/2}\big)\nabla k_\varepsilon \cdot \nabla z + \int_\Omega k_\varepsilon^{3/2} z \\ &\quad = \int_\Omega \big(\varepsilon + [k_\varepsilon]_\varepsilon\big)^{1/2}|\nabla u_\varepsilon|^2 z \quad \text{for a.e. } t \in [0,T],\ \forall z \in W^{1,2}(\Omega) \end{aligned}\right\} \tag{12}$$

and $u_\varepsilon(0) = u_0$, $k_\varepsilon(0) = k_0$ a.e. in Ω.

This result can be deduced from [11, Chap. 3, 1.4, Théorème 1.2]. For details of this reasoning we refer to the proof of the proposition in [12, pp. 1889–1897].

2° *A priori estimates on* $u_\varepsilon, k_\varepsilon$. In (11) and (12), we write s in place of t, insert $v = u(\cdot,s)$ into (11), $z = 1$ into (12) and integrate over the interval $[0,t]$ $(0 < t \leqslant T)$. This gives

$$\left.\begin{aligned} &\max_{t\in[0,T]} \int_\Omega u_\varepsilon^2(x,t)\,\mathrm{d}x + \int_Q \big(\big(\varepsilon + [k_\varepsilon]_\varepsilon\big)^{1/2} + \varepsilon|\nabla u_\varepsilon|^2\big)|\nabla u_\varepsilon|^2 \leqslant \frac{3}{2}\int_\Omega u_0^2, \\ &\max_{t\in[0,T]} \int_\Omega k_\varepsilon(x,t)\,\mathrm{d}x + \int_Q k_\varepsilon^{3/2} \leqslant \int_\Omega \Big(\frac{3}{2}u_0^2 + k_0\Big), \end{aligned}\right\} \tag{13}$$

$$\int_\Omega \Big(\frac{1}{2}u_\varepsilon^2(x,t) + k_\varepsilon(x,t)\Big)\,\mathrm{d}x + \int_0^t \int_\Omega k_\varepsilon^{3/2} \leqslant \int_\Omega \Big(\frac{1}{2}u_0^2 + k_0\Big) \quad \text{for a.e. } t \in [0,T]. \tag{14}$$

We take $z = \frac{1}{(\varepsilon + [k_\varepsilon(\cdot,s)]_\varepsilon)^{1/2}}$ in (12) and obtain, for a.e. $t \in [0,T]$

$$\int_\Omega \Big(\int_0^{k_0(x)} \frac{d\sigma}{(\varepsilon + [\sigma]_\varepsilon)^{1/2}}\Big)\,\mathrm{d}x + \int_0^t \int_\Omega |\nabla u_\varepsilon|^2 \leqslant 2\int_\Omega k_\varepsilon^{1/2}(x,t)\,\mathrm{d}x + \int_0^t \int_\Omega k_\varepsilon\big(1 + \varepsilon^{1/2}(\varepsilon + k_\varepsilon)^{1/2}\big). \tag{15}$$

Next, given $0 < \delta < 1$, we insert $z = 1 - \frac{1}{(1+k_\varepsilon(\cdot,s))^\delta}$ into (12). An elementary but lengthy calculation gives:

$$\mu \int_Q \frac{|\nabla k_\varepsilon|^2}{(1 + k_\varepsilon)^{1+\delta}} \leqslant c_1, \qquad \int_Q \big(k_\varepsilon^{3r/2} + |\nabla k_\varepsilon|^r\big) \leqslant c_2 \quad \forall r \in \Big[1, \frac{4}{3}\Big[, \tag{16}$$

where $c_1 = c_1(\delta) \to +\infty$ as $\delta \to 0$ or $\delta \to 1$, respectively, and $c_2 = c_2(r) \to +\infty$ as $r \to \frac{4}{3}$.

We derive an integral estimate on $[k_\varepsilon]_\varepsilon^{1/2}\nabla k_\varepsilon$. To this end, define $\Phi_\varepsilon(\xi) := \int_0^\xi [\sigma]_\varepsilon^{1/2}\,\mathrm{d}\sigma$, $0 \leqslant \xi < +\infty$. Given $0 < \gamma < 1$, we test (12) by $z = 1 - \frac{1}{[1+\Phi_\varepsilon(k_\varepsilon(\cdot,s))]^\gamma}$ (cf. [7]). Analogously as above, we obtain:

$$\int_Q \big|[k_\varepsilon]_\varepsilon^{1/2}\nabla k_\varepsilon\big|^{\tilde{s}} \leqslant c_3 \quad \forall \tilde{s} \in \Big[1, \frac{8}{7}\Big[, \qquad \int_Q k_\varepsilon^q \leqslant c_4 \quad \forall q \in \Big[1, \frac{16}{7}\Big[, \tag{17}$$

where $c_3 = c_3(\tilde{s}) \to +\infty$ as $\tilde{s} \to \frac{8}{7}$, and $c_4 = c_4(q) \to +\infty$ as $q \to \frac{16}{7}$ (notice that the estimate on k_ε^q follows by combining (13), (16) with the estimate on $[k_\varepsilon]_\varepsilon^{1/2}\nabla k_\varepsilon$ and an elementary interpolation property of the spaces $L^\rho(0,T;L^\sigma(\Omega))$).

3° *A priori estimates on* $u'_\varepsilon, k'_\varepsilon$. We have, for all $0 < \varepsilon \leqslant 1$,

$$\|u'_\varepsilon\|_{L^{4/3}(0,T;W^{-1,4/3})} \leqslant c_5, \qquad \|k'_\varepsilon\|_{L^1(0,T;W^{-1,s'})} \leqslant c_6 \quad \forall s \in]8, +\infty[, \tag{18}$$

where $c_6 = c_6(s) \to +\infty$ as $s \to 8$. Indeed, the estimate on u'_ε can be easily deduced from (11) and (13), whereas the estimate on k'_ε follows from (12), (13), (16) and (17). We notice that in this reasoning the derivatives u'_ε resp. k'_ε are understood in the sense of distributions from $]0,T[$ into $W^{-1,4/3}(\Omega)$ resp. $W^{-1,s'}(\Omega)$.

4° *Passage to the limit* $\varepsilon \to 0$. First, we notice that (16) and the estimate on k'_ε in (18) imply the existence of a subsequence of (k_ε) (not relabeled) such that:

$$k_\varepsilon \to k \quad \text{strongly in } L^r\big(0,T;L^2(\Omega)\big) \left(1 < r < \frac{4}{3}\right) \text{ as } \varepsilon \to 0. \tag{19}$$

Indeed, we have $W^{1,r}(\Omega) \subset L^2(\Omega)$ compactly. Identifying $L^2(\Omega)$ with its dual $(L^2(\Omega))^*$, it follows that $L^2(\Omega) \subset W^{-1,s'}(\Omega)$ $(8 < s < +\infty)$ continuously. The compactness result [19, Corollary 4] may therefore be applied to $X = W^{1,r}(\Omega)$, $B = L^2(\Omega)$, $Y = W^{-1,s'}(\Omega)$ and the family of mappings $\{k_\varepsilon;\ 0 < \varepsilon \leqslant 1\}$. Whence (19).

From (13), (15), (16), (17) and the estimate on u'_ε in (18) we obtain (again by passing to subsequences if necessary):

$$\left.\begin{array}{l} u_\varepsilon \to u \quad \text{weakly in } L^2\big(0,T;W_0^{1,2}(\Omega)\big),\ \text{weakly}^* \text{ in } L^\infty\big(0,T;L^2(\Omega)\big), \\ u'_\varepsilon \to u' \quad \text{weakly in } L^{4/3}\big(0,T;W^{-1,4/3}(\Omega)\big), \\ u_\varepsilon \to u \quad \text{strongly in } L^2\big(0,T;L^2(\Omega)\big)\ \big(\text{cf. [11, Chap. 1, 5.2], [19, Corollary 4]}\big), \end{array}\right\} \tag{20}$$

$$\left.\begin{array}{l} k_\varepsilon \to k \quad \text{weakly in } L^q(Q) \left(1 < q < \dfrac{16}{7}\right),\ \text{weakly in } L^r\big(0,T;W^{1,r}(\Omega)\big) \left(1 < r < \dfrac{4}{3}\right), \\ \big(\varepsilon + [k_\varepsilon]_\varepsilon\big)^{1/4} \nabla u_\varepsilon \to k^{1/4}\nabla u \quad \text{weakly in } \big[L^2(Q)\big]^2, \\ \dfrac{\nabla k_\varepsilon}{(1+k_\varepsilon)^{(1+\delta)/2}} \to \dfrac{\nabla k}{(1+k)^{(1+\delta)/2}} \quad \text{weakly in } \big[L^2(Q)\big]^2 \text{ for any } 0 < \delta < 1, \\ [k_\varepsilon]_\varepsilon^{1/2}\nabla k_\varepsilon \to \dfrac{2}{3}\nabla k^{3/2} \quad \text{weakly in } \big[L^{\tilde s}(Q)\big]^2 \left(1 < \tilde s < \dfrac{8}{7}\right) \end{array}\right\} \tag{21}$$

as $\varepsilon \to 0$. Then the properties of u and k stated in (3), are readily seen. The passage to the limit $\varepsilon \to 0$ in (14) and (15) is easily carried out by using standard arguments from integration theory.

Let $v \in L^4(0,T;W_0^{1,4}(\Omega))$. We insert $v = v(\cdot,t)$ into (11), integrate over $[0,T]$ and carry out the passage to the limit $\varepsilon \to 0$. With the help of (20) and (21), we find (6). Since $k \in L^\infty(0,T;L^1(\Omega))$, we have indeed $u' \in L^2(0,T;W^{-1,4/3}(\Omega))$. By a routine argument, $u(0) = u_0$ a.e. in Ω.

The basic problem of the passage to the limit $\varepsilon \to 0$ in (12) is embodied in the term $(\varepsilon + [k_\varepsilon]_\varepsilon)^{1/2}|\nabla u_\varepsilon|^2$. To obtain its L^1-convergence when $\varepsilon \to 0$, we write (6) in the form:

$$\int_0^T \big\langle u'(t), v(t)\big\rangle_{W^{-1,4/3},W_0^{1,4}} \,\mathrm{d}t + \int_Q (a+b)\nabla u \cdot \nabla v = 0 \quad \forall v \in L^4\big(0,T;W_0^{1,4}(\Omega)\big), \tag{22}$$

where $a := k^{1/2} - (1+k)^{1/2}$, $b := (1+k)^{1/2}$ a.e. in Q. Observing that $\frac{\nabla k}{(1+k)^{3/4}} \in [L^2(Q)]^2$ ($\delta = \frac{1}{2}$ in (5)), we obtain

$$k^{1/2} = a + b, \quad -1 \leqslant a \leqslant 0 \text{ a.e. in } Q, \quad b^2 \in L^\infty\big(0,T;L^1(\Omega)\big),\ \nabla b^{1/2} \in \big[L^2(Q)\big]^2.$$

Then from (22) it follows that:

$$\frac{1}{2}\int_\Omega u^2(x,t)z(x)\,\mathrm{d}x + \int_0^t \int_\Omega (a+b)\big(|\nabla u|^2 z + u\nabla u \cdot \nabla z\big) = \frac{1}{2}\int_\Omega u_0^2 z \quad \text{for a.e. } t \in [0,T],\ \forall z \in W_0^{1,4}(\Omega), \tag{23}$$

i.e. (8) holds. We remark that the representation $k^{1/2} = (k^{1/2} - (1+k)^{1/2}) + (1+k)^{1/2}$ has been already used in [14]. We notice that the local energy equality (23) holds if a and b in (22) are measurable functions in Q such that

$$|a| \leqslant C = \text{const}, \quad b \geqslant 0 \text{ a.e. in } Q, \quad b^2 \in L^\infty\big(0,T;L^1(\Omega)\big),\ \nabla b^{1/2} \in \big[L^2(Q)\big]^n, \quad \int_Q b|\nabla u|^2 < +\infty$$

$(n = 2$ or $n = 3)$. The proof of this result will be the object of a forthcoming paper.

Let $z \in W_0^{1,4}(\Omega)$, $z \geqslant 0$ a.e. in Ω. We insert $v = u_\varepsilon(\cdot,t)z$ into (11). With the help of (20) and (21) we obtain:

$$\limsup_{\varepsilon\to 0} \int_Q \big(\varepsilon + [k_\varepsilon]_\varepsilon\big)^{1/2}|\nabla u_\varepsilon|^2 z \leqslant -\frac{1}{2}\int_\Omega u^2(x,T)z(x)\,\mathrm{d}x + \frac{1}{2}\int_\Omega u_0^2 z - \int_Q k^{1/2}u\nabla u \cdot \nabla z \overset{(8)}{=} \int_Q k^{1/2}|\nabla u|^2 z$$

$$\leqslant \liminf_{\varepsilon\to 0} \int_Q \big(\varepsilon + [k_\varepsilon]_\varepsilon\big)^{1/2}|\nabla u_\varepsilon|^2 z.$$

Hence, by routine arguments,

$$\lim_{\varepsilon\to 0}\int_Q \left(\varepsilon+[k_\varepsilon]_\varepsilon\right)^{1/2}|\nabla u_\varepsilon|^2 z\alpha=\int_Q k^{1/2}|\nabla u|^2 z\alpha \quad \forall z\in W_0^{1,4}(\Omega),\ \forall\alpha\in L^\infty(Q). \tag{24}$$

We now insert $z\in W_0^{1,s}(\Omega)$ $(8<s<+\infty)$ into (12), multiply by $\alpha\in C^1([0,T])$, $\alpha(T)=0$, integrate over $[0,T]$, integrate by parts the term involving k'_ε and then let $\varepsilon\to 0$. Observing (24) we find:

$$-\int_Q kz\alpha'+\int_Q\left(\mu\nabla k+\frac{2}{3}\nabla k^{3/2}\right)\cdot\nabla z\alpha+\int_Q k^{3/2}z\alpha=\int_\Omega k_0 z\alpha(0)+\int_Q k^{1/2}|\nabla u|^2 z\alpha. \tag{25}$$

To establish the existence of $k'\in L^1(0,T;W^{-1,s'}(\Omega))$, we define $F:]0,T[\to W^{-1,s'}(\Omega)$ $(8<s<+\infty)$ by

$$\langle F(t),z\rangle_{W^{-1,s'},W_0^{1,s}}:=-\int_\Omega\left(\mu\nabla k(t)+\frac{2}{3}\nabla k^{3/2}(t)\right)\cdot\nabla z+\int_\Omega\left(-k^{3/2}(t)+k^{1/2}(t)\left|\nabla u(t)\right|^2\right)z,\quad z\in W_0^{1,s}(\Omega).$$

By a well-known theorem of Pettis, the mapping F is measurable. It is readily seen that $F\in L^1(0,T;W^{-1,s'}(\Omega))$. Hence, for any $\alpha\in C_c^1(]0,T[)$, (25) can be equivalently written in the form:

$$\left\langle-\int_0^T k(t)\alpha'(t)\,\mathrm{d}t,z\right\rangle_{W^{-1,s'},W_0^{1,s}}=\left\langle\int_0^T F(t)\alpha(t)\,\mathrm{d}t,z\right\rangle_{W^{-1,s'},W_0^{1,s}},\quad z\in W_0^{1,s}(\Omega).$$

From [3, Appendix, Proposition A.6], we conclude that there exists $k'\in L^1(0,T;W^{-1,s'}(\Omega))$. Thus,

$$\langle k'(t),z\rangle_{W^{-1,s'},W_0^{1,s}}=\langle F(t),z\rangle_{W^{-1,s'},W_0^{1,s}}\quad\text{for a.e. } t\in[0,T],\ \forall z\in W_0^{1,s}(\Omega), \tag{26}$$

where the Lebesgue null set of those $t\in[0,T]$ for which (26) fails, does not depend on z.

To finish the sketch of proof, let $w\in L^\infty(0,T;W_0^{1,s}(\Omega))$ $(8<s<+\infty)$. Inserting $z=w(\cdot,t)$ into (26) and integrating over $[0,T]$, we obtain (7). Finally, we multiply (26) by $\alpha\in C^1([0,T])$, $\alpha(T)=0$ and $\alpha(0)=1$, integrate over $[0,T]$ and integrate by parts the term involving k'. Comparing the result with (25) we arrive at:

$$\langle k(0),z\rangle_{W^{-1,s'},W_0^{1,s}}=\int_\Omega k_0 z\quad\forall z\in W_0^{1,s}(\Omega).$$

References

[1] G.K. Batchelor, An Introduction to Fluid Mechanics, Cambridge University Press, Cambridge, 1967.
[2] N. Bourbaki, Éléments de mathématique. Livre VI. Intégration 1–4, Hermann, Paris, 1965.
[3] H. Brézis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, North-Holland Publ. Comp, Amsterdam, 1973.
[4] M. Bulíček, R. Lewandowski, J. Málek, On evolutionary Navier–Stokes–Fourier type systems in three spatial dimensions, Comment. Math. Univ. Carol. 52 (2011) 89–114.
[5] J. Cousteix, Aérodynamique. Turbulence et couche limite, Cepadues-Éditions, Toulouse, 1989.
[6] J. Droniou, Intégration et espaces de Sobolev à valeurs vectorielles, http://www-gm3.univ-mrs.fr/polys/.
[7] P.-E. Druet, J. Naumann, On the existence of weak solutions to a stationary one-equation RANS model with unbounded eddy viscosities, Ann. Univ. Ferrara 55 (2009) 67–87.
[8] T. Gallouët, J. Lederer, R. Lewandowski, F. Murat, L. Tartar, On a turbulent system with unbounded viscosities, Nonlinear Anal. 52 (2003) 1051–1068.
[9] R. Lewandowski, The mathematical analysis of the coupling of a turbulent kinetic energy equation to the Navier–Stokes equation with an eddy viscosity, Nonlinear Anal. 28 (1997) 393–417.
[10] R. Lewandowski, Analyse mathématique et océanographie, Masson, Paris, 1997.
[11] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Gauthier-Villars, Paris, 1969.
[12] J. Naumann, On the existence of weak solutions to the equations of non-stationary motion of heat-conducting incompressible viscous fluids, Math. Methods Appl. Sci. 29 (2006) 1883–1906.
[13] J. Naumann, On weak solutions to the equations of non-stationary motion of heat-conducting incompressible viscous fluids: Defect measure and energy equality, in: Parabolic and Navier–Stokes Equs., Warsaw, in: Banach Center Publ., vol. 81, 2008, pp. 287–296.
[14] J. Naumann, J. Wolf, Existence of weak solutions to a simplified steady system of turbulence modeling, C. R. Acad. Sci. Paris, Ser. I 350 (2012) 45–50.
[15] J. Naumann, J. Wolf, On Prandtl's turbulence model: Existence of weak solutions to the equations of stationary turbulent pipe-flow, Discrete Contin. Dyn. Syst., Ser. S 6 (2013) 1371–1390.
[16] H. Oertel, Prandtl-essentials in Fluid Mechanics, 3rd edition, Springer, New York, 2010.
[17] S.B. Pope, Turbulent Flows, Cambridge University Press, Cambridge, 2006.
[18] L. Prandtl, Über ein neues Formelsystem für die ausgebildete Turbulenz, Nachr. Akad. Wiss. Gött. Math.-Phys. Kl. 1 (1946) 6–19.
[19] J. Simon, Compact sets in the spaces $L^p(0,T;B)$, Ann. Mat. Pura Appl. 146 (1987) 65–96.
[20] D.C. Wilcox, Turbulence Modeling for CFD, DCW Industries, La Cañada, 2006.