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Complex Analysis

A comparison principle for the log canonical threshold

Un principe de comparaison pour le seuil log-canonique

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ABSTRACT

In this note, we prove a comparison principle for the log canonical threshold of plurisubharmonic functions.

RÉSUMÉ

Dans cette note, nous démontrons un principe de comparaison pour le seuil log-canonique des fonctions plurisousharmoniques.

1. Introduction

Let Ω be a domain in $\mathbb{C}^n$ and $\varphi \in \mathrm{PSH}(\Omega)$. Following Demailly and Kollár [5], we introduce the log canonical threshold of φ at 0:

$$c_\varphi(0) = \sup\left\{c > 0\colon\ e^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } 0\right\}.$$

It is an invariant of the singularity of φ at 0. We refer to [3,2,4–6,8] for further information about this number.

The main result is the following theorem:

Theorem 1.1. *Let Ω be a domain in $\mathbb{C}^n$ and $\{\Omega_j\}_{\{j\geqslant 1\}}$ be a sequence of smooth domains such that $\Omega \Supset \Omega_1 \Supset \Omega_2 \Supset \ldots$ and $\bigcap_{j=1}^{\infty} \Omega_j = \{0\}$. Assume that $u, v \in \mathrm{PSH}(\Omega)$. If $u \geqslant v$ on $\partial\Omega_j$ for all $j \geqslant 1$, then $c_u(0) \geqslant c_v(0)$.*

2. Proof of the main result

First, we need the following lemma which follows from Proposition 1.5 and Theorem 4.2 in [5]:

Lemma 2.1. *Let $u \in \mathrm{PSH}^-(\Omega)$. Then*

$$\lim_{j\to\infty} c_{\max(u,\, j\log\|z\|)}(0) = c_u(0).$$

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Proof. We include a detailed proof for the reader's convenience. In a first step, we prove the lemma in the case $u = \max(\log|f_1|, \dots, \log|f_N|)$, where $f_1, \dots, f_N$ are germs of holomorphic functions at 0. Without loss of the generality, we can assume that the degrees of $f_1, \dots, f_N$ are large enough. By Proposition 1.5 in [5], there exist $a_1, \dots, a_N, b_1, \dots, b_n \in \mathbb{C}$ such that:

$$c_{\max(u, j\log\|z\|)}(0) = c_{\log|a_1 f_1 + \dots + a_N f_N + b_1 z_1^j + \dots + b_n z_n^j|}(0).$$

On the other hand, by Theorem 2.9 in [5], we have:

$$\left|c_{\log|a_1 f_1 + \dots + a_N f_N + b_1 z_1^j + \dots + b_n z_n^j|}(0) - c_{\log|a_1 f_1 + \dots + a_N f_N|}(0)\right| \leqslant c_{\log|b_1 z_1^j + \dots + b_n z_n^j|}(0) = \frac{n}{j}.$$

Therefore

$$c_{\max(u, j\log\|z\|)}(0) \leqslant c_{\log|a_1 f_1 + \dots + a_N f_N|}(0) + \frac{n}{j} \leqslant c_u(0) + \frac{n}{j}.$$

This implies that:

$$\lim_{j\to\infty} c_{\max(u, j\log\|z\|)}(0) = c_u(0).$$

The final step consists of reducing the proof of the lemma to the case $u = \log(|f_1|^2 + \dots + |f_N|^2)$, with $f_1, \dots, f_N$ are germs of holomorphic functions at 0. Let $\mathcal{H}_{mu}(\Omega)$ be the Hilbert space of holomorphic functions f on Ω such that

$$\int_\Omega |f|^2 e^{-2mu}\,\mathrm{d}V < +\infty,$$

and let $\psi_m = \frac{1}{2m}\log\sum|g_{m,k}|^2$ where $\{g_{m,k}\}_{k\geqslant 1}$ is an orthonormal basis of $\mathcal{H}_{mu}(\Omega)$. By Theorem 4.2 in [5], there are constants $C_1, C_2 > 0$ independent of m such that:

$$u(z) - \frac{C_1}{m} \leqslant \psi_m(z) \leqslant \sup_{|\zeta - z|<r} u(\zeta) + \frac{1}{m}\log\frac{C_2}{r^n} \tag{1}$$

for every $z \in \Omega$ and $r < d(z, \partial\Omega)$ and

$$\frac{1}{c_u} - \frac{1}{m} \leqslant \frac{1}{c_{\psi_m}} \leqslant \frac{1}{c_u}. \tag{2}$$

Since $\max(u, j\log|z|) \leqslant \max(\psi_m, j\log|z|) + \frac{C_1}{m}$, we have:

$$c_{\max(u, j\log\|z\|)}(0) \leqslant c_{\max(\psi_m, j\log|z|)}(0), \quad \forall j, m \geqslant 1.$$

Letting $j \to \infty$, by the first step, we get:

$$\lim_{j\to\infty} c_{\max(u, j\log\|z\|)}(0) \leqslant \lim_{j\to\infty} c_{\max(\psi_m, j\log|z|)}(0) = c_{\psi_m}(0), \quad \forall m \geqslant 1. \tag{3}$$

Thanks to (1) and (3), we obtain:

$$\lim_{j\to\infty} c_{\max(u, j\log\|z\|)}(0) = c_u(0). \qquad \square$$

2.1. Proof of the main theorem

Without loss of the generality, we can assume that Ω is the unit ball. By Lemma 2.1, we only have to prove the main theorem in the case $u, v \in \mathrm{PSH} \cap L^\infty_{loc}(\Omega \setminus \{0\})$ and $u|_{\partial\Omega} = v|_{\partial\Omega} = 0$. For each $j \geqslant 1$, we set:

$$\tilde{u}_j = \sup\{\varphi \in \mathrm{PSH}^-(\Omega):\ \varphi \leqslant u \text{ on } \Omega_j\}.$$

Then $\tilde{u}_j \geqslant u$ on Ω, $\tilde{u}_j = u$ on $\overline{\Omega}_j$ and $\tilde{u}_j \leqslant \tilde{u}_{j+1}$ on Ω. Set $\tilde{u} = (\lim_{j\to\infty}\tilde{u}_j)^* \in \mathrm{PSH}^- \cap L^\infty_{loc}(\Omega \setminus \{0\})$. By Kolodziej's theorem [7], there exists $\phi_j \in \mathrm{PSH}^- \cap L^\infty(\Omega)$ such that $\phi_j|_{\partial\Omega} = 0$ and $(dd^c\phi_j)^n = 1_{\Omega\setminus\Omega_{j+1}}(dd^c u)^n$, where 1_E is the characterization function of E. By the comparison principle, we get $\phi_j \searrow \phi$ on Ω with $(dd^c\phi)^n = 1_{\Omega\setminus\{0\}}(dd^c u)^n$. Using the comparison principle for $\tilde{u}_j + \phi_j, u$ on the set $\Omega \setminus \Omega_{j+1}$, it follows that $\tilde{u}_j + \phi_j \leqslant u$ on $\Omega \setminus \Omega_{j+1}$. Hence $\tilde{u} + \phi \leqslant u$. On the other hand, from Corollary 5.7 in [1] and from $\int_{\{0\}}(dd^c\phi)^n = 0$, we get $c_\phi(0) = 0$. Thus by the Hölder inequality, we obtain $c_u(0) = c_{\tilde{u}}(0)$.

Similarly, we set:

$$\tilde{v}_j = \sup\{\varphi \in \mathrm{PSH}^-(\Omega)\colon \varphi \leqslant v \text{ on } \Omega_j\},$$

and

$$\tilde{v} = \Big(\lim_{j\to\infty} \tilde{v}_j\Big)^*.$$

We will prove that $\tilde{u} \geqslant \tilde{v}$. Indeed, set $w_j = \tilde{u}_j$ if $z \in \overline{\Omega}_j$ and $w_j = \max(\tilde{u}_j, \tilde{v}_j)$ if $z \in \Omega \setminus \Omega_j$. We have $w_j \in \mathrm{PSH}^-(\Omega)$ and $w_j = u$ on Ω_j. By the definition of $\tilde{u}_j$ we get $\tilde{u}_j \geqslant w_j$. Hence $\tilde{u}_j \geqslant \tilde{v}_j$ on $\Omega \setminus \Omega_j$. Letting $j \to \infty$, we get $\tilde{u} \geqslant \tilde{v}$. This implies that $c_u(0) \geqslant c_v(0)$.

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