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Harmonic Analysis

The John–Nirenberg inequality with sharp constants

Meilleures constantes dans l'inégalité de John–Nirenberg

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ABSTRACT

We consider the one-dimensional John–Nirenberg inequality:

$$\left|\left\{x \in I_0\colon \left|f(x)-f_{I_0}\right|>\alpha\right\}\right| \leqslant C_1|I_0|\exp\left(-\frac{C_2}{\|f\|_*}\alpha\right).$$

A. Korenovskii found that the sharp C_2 here is $C_2=2/\mathrm{e}$. It is shown in this paper that if $C_2=2/\mathrm{e}$, then the best possible C_1 is $C_1=\frac{1}{2}\mathrm{e}^{4/\mathrm{e}}$.

RÉSUMÉ

On considère l'inégalité de John–Nirenberg unidimensionnelle :

$$\left|\left\{x \in I_0\colon \left|f(x)-f_{I_0}\right|>\alpha\right\}\right| \leqslant C_1|I_0|\exp\left(-\frac{C_2}{\|f\|_*}\alpha\right).$$

A. Korenovskii a montré que la meilleure constante C_2 était égale à $2/\mathrm{e}$. Dans cette Note, on montre que si $C_2=2/\mathrm{e}$, alors la meilleure constante possible pour C_1 est $C_1=\frac{1}{2}\mathrm{e}^{4/\mathrm{e}}$.

1. Introduction

Let $I_0 \subset \mathbb{R}$ be an interval and let f be an integrable function on I_0. Given a measurable set $E \subset \mathbb{R}$, denote by $|E|$ its Lebesgue measure. Given a subinterval $I \subset I_0$, set $f_I = \frac{1}{|I|}\int_I f$ and

$$\Omega(f;I) = \frac{1}{|I|}\int_I \left|f(x)-f_I\right|\,\mathrm{d}x.$$

We say that $f \in BMO(I_0)$ if $\|f\|_* \equiv \sup_{I\subset I_0}\Omega(f;I) < \infty$. The classical John–Nirenberg inequality [1] says that there are $C_1, C_2 > 0$ such that for any $f \in BMO(I_0)$,

$$\left|\left\{x \in I_0\colon \left|f(x)-f_{I_0}\right|>\alpha\right\}\right| \leqslant C_1|I_0|\exp\left(-\frac{C_2}{\|f\|_*}\alpha\right) \quad (\alpha > 0).$$

A. Korenovskii [4] (see also [5, p. 77]) found the best possible constant C_2 in this inequality, namely, he showed that $C_2 = 2/\mathrm{e}$:

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$$\left|\left\{x \in I_0\colon \left|f(x) - f_{I_0}\right| > \alpha\right\}\right| \leqslant C_1 |I_0| \exp\left(-\frac{2/\mathrm{e}}{\|f\|_*}\alpha\right) \quad (\alpha > 0), \tag{1.1}$$

and in general the constant $2/\mathrm{e}$ here cannot be increased.

A question about the sharp C_1 in (1.1) remained open. In [4], (1.1) was proved with $C_1 = \mathrm{e}^{1+2/\mathrm{e}} = 5.67323\ldots$. The method of the proof in [4] was based on the Riesz sunrise lemma and on the use of non-increasing rearrangements. In this paper, we give a different proof of (1.1), yielding the sharp constant $C_1 = \frac{1}{2}\mathrm{e}^{4/\mathrm{e}} = 2.17792\ldots$.

Theorem 1.1. *Inequality (1.1) holds with $C_1 = \frac{1}{2}\mathrm{e}^{4/\mathrm{e}}$, and this constant is the best possible.*

We also use as the main tool the Riesz sunrise lemma. But instead of the rearrangement inequalities, we obtain a direct pointwise estimate for any *BMO*-function (see Theorem 2.2 below). The proof of this result is inspired (and close in spirit) by a recent decomposition of an arbitrary measurable function in terms of mean oscillations (see [2,6]).

We mention several recent papers [7,8] where sharp constants in some different John–Nirenberg-type estimates were found by means of the Bellman function method.

2. Proof of Theorem 1.1

We shall use the following version of the Riesz sunrise lemma [3].

Lemma 2.1. *Let g be an integrable function on some interval $I_0 \subset \mathbb{R}$, and suppose $g_{I_0} \leqslant \alpha$. Then there is at most countable family of pairwise disjoint subintervals $I_j \subset I_0$ such that $g_{I_j} = \alpha$, and $g(x) \leqslant \alpha$ for almost all $x \in I_0 \setminus (\bigcup_j I_j)$.*

Observe that the family $\{I_j\}$ in Lemma 2.1 may be empty if $g(x) < \alpha$ a.e. on I_0.

Theorem 2.2. *Let $f \in BMO(I_0)$, and let $0 < \gamma < 1$. Then there is at most countable decreasing sequence of measurable sets $G_k \subset I_0$ such that $|G_k| \leqslant \min(2\gamma^k, 1)|I_0|$ and for a.e. $x \in I_0$,*

$$\left|f(x) - f_{I_0}\right| \leqslant \frac{\|f\|_*}{2\gamma} \sum_{k=0}^{\infty} \chi_{G_k}(x). \tag{2.1}$$

Proof. Given an interval $I \subseteq I_0$, set $E(I) = \{x \in I\colon\ f(x) > f_I\}$. Let us show that there is at most a countable family of pairwise disjoint subintervals $I_j \subset I_0$ such that $\sum_j |I_j| \leqslant \gamma |I_0|$ and for a.e. $x \in I_0$,

$$(f - f_{I_0})\chi_{E(I_0)} \leqslant \frac{\|f\|_*}{2\gamma}\chi_{E(I_0)} + \sum_j (f - f_{I_j})\chi_{E(I_j)}. \tag{2.2}$$

We apply Lemma 2.1 with $g = f - f_{I_0}$ and $\alpha = \frac{\|f\|_*}{2\gamma}$. One can assume that $\alpha > 0$ and the family of intervals $\{I_j\}$ from Lemma 2.1 is non-empty (since otherwise (2.2) holds trivially only with the first term on the right-hand side). Since $g_{I_j} = \alpha$, we obtain:

$$\sum_j |I_j| = \frac{1}{\alpha} \int_{\bigcup_j I_j} (f - f_{I_0})\,\mathrm{d}x \leqslant \frac{1}{\alpha} \int_{\{x \in I_0\colon f(x) > f_{I_0}\}} (f - f_{I_0})\,\mathrm{d}x$$
$$= \frac{1}{2\alpha}\Omega(f; I_0)|I_0| \leqslant \gamma |I_0|.$$

Since $g_{I_j} = \alpha$, we have $f_{I_j} = f_{I_0} + \alpha$, and hence:

$$f - f_{I_0} = (f - f_{I_0})\chi_{I_0 \setminus \bigcup_j I_j} + \alpha\chi_{\bigcup_j I_j} + \sum_j (f - f_{I_j})\chi_{I_j}.$$

This proves (2.2) since $f - f_{I_0} \leqslant \alpha$ a.e. on $I_0 \setminus \bigcup_j I_j$.

The sum on the right-hand side of (2.2) consists of the terms of the same form as the left-hand side. Therefore, one can proceed iterating (2.2). Denote $I_j^1 = I_j$, and let I_j^k be the intervals obtained after the k-th step of the process. Iterating (2.2) m times yields:

$$(f - f_{I_0})\chi_{E(I_0)} \leqslant \frac{\|f\|_*}{2\gamma} \sum_{k=0}^{m} \sum_j \chi_{E(I_j^k)}(x) + \sum_i (f - f_{I_i^{m+1}})\chi_{E(I_i^{m+1})}$$

(where $I_j^0 = I_0$). If there is m such that for any i each term of the second sum is bounded trivially by $\frac{\|f\|_*}{2\gamma}\chi_{E(I_i^{m+1})}$, we stop the process, and we would obtain the finite sum with respect to k. Otherwise, let $m \to \infty$. Using that

$$\left|\bigcup_i I_i^{m+1}\right| \leqslant \gamma\left|\bigcup_l I_l^m\right| \leqslant \cdots \leqslant \gamma^{m+1}|I_0|,$$

we get that the support of the second term will tend to a null set. Hence, setting $E_k = \bigcup_j E(I_j^k)$, for a.e. $x \in E(I_0)$ we obtain:

$$(f - f_{I_0})\chi_{E(I_0)} \leqslant \frac{\|f\|_*}{2\gamma}\left(\chi_{E(I_0)}(x) + \sum_{k=1}^{\infty}\chi_{E_k}(x)\right). \tag{2.3}$$

Observe that $E(I_j) = \{x \in I_j\colon\ f(x) > f_{I_0} + \alpha\} \subset E(I_0)$. From this and from the above process we easily get that $E_{k+1} \subset E_k$. Also, $E_k \subset \bigcup_j I_j^k$, and hence $|E_k| \leqslant \gamma^k|I_0|$.

Setting now $F(I) = \{x \in I\colon\ f(x) \leqslant f_I\}$, and applying the same argument to $(f_{I_0} - f)\chi_{F(I)}$, we obtain:

$$(f_{I_0} - f)\chi_{F(I_0)} \leqslant \frac{\|f\|_*}{2\gamma}\left(\chi_{F(I_0)}(x) + \sum_{k=1}^{\infty}\chi_{F_k}(x)\right), \tag{2.4}$$

where $F_{k+1} \subset F_k$ and $|F_k| \leqslant \gamma^k|I_0|$. Also, $F_k \cap E_k = \emptyset$. Therefore, summing (2.3) and (2.4) and setting $G_0 = I_0$ and $G_k = E_k \cup F_k$, $k \geqslant 1$, we get (2.1). $\square$

Proof of Theorem 1.1. Let us show first that the best possible C_1 in (1.1) satisfies $C_1 \geqslant \frac{1}{2}e^{4/e}$. It suffices to give an example of f on I_0 such that for any $\varepsilon > 0$,

$$\left|\left\{x \in I_0\colon\ \left|f(x) - f_{I_0}\right| > 2(1-\varepsilon)\|f\|_*\right\}\right| = |I_0|/2. \tag{2.5}$$

Let $I_0 = [0, 1]$ and take $f = \chi_{[0,1/4]} - \chi_{[3/4,1]}$. Then $f_{I_0} = 0$. Hence, (2.5) would follow from $\|f\|_* = 1/2$. To show the latter fact, take an arbitrary $I \subset I_0$. It is easy to see that computations reduce to the following cases: I contains only $1/4$ and I contains both $1/4$ and $3/4$.

Assume that $I = (a, b)$, $1/4 \in I$, and $b < 3/4$. Let $\alpha = \frac{1}{4} - a$ and $\beta = b - \frac{1}{4}$. Then $f_I = \alpha/(\alpha+\beta)$ and:

$$\Omega(f; I) = \frac{2}{\alpha+\beta}\int_{\{x \in I\colon\ f > f_I\}}(f - f_I) = \frac{2\alpha\beta}{(\alpha+\beta)^2} \leqslant 1/2$$

with $\Omega(f; I) = 1/2$ if $\alpha = \beta$.

Consider the second case. Let $I = (a, b)$, $a < 1/4$ and $b > 3/4$. Let α be as above and $\beta = b - \frac{3}{4}$. Then:

$$\Omega(f; I) = \frac{2}{\alpha+\beta+1/2}\int_{\{x \in I\colon\ f > f_I\}}(f - f_I) = \frac{4\alpha(4\beta+1)}{(2\alpha+2\beta+1)^2}.$$

Since

$$\sup_{0 \leqslant \alpha,\beta \leqslant 1/4}\frac{4\alpha(4\beta+1)}{(2\alpha+2\beta+1)^2} = 1/2,$$

this proves that $\|f\|_* = 1/2$. Therefore, $C_1 \geqslant \frac{1}{2}e^{4/e}$. Let us show now the converse inequality.

Let $f \in BMO(I_0)$. Setting $\psi(x) = \sum_{k=0}^{\infty}\chi_{G_k}(x)$, where G_k are from Theorem 2.2, we have:

$$\begin{aligned}\left|\left\{x \in I_0\colon\ \psi(x) > \alpha\right\}\right| &= \sum_{k=0}^{\infty}|G_k|\chi_{[k,k+1)}(\alpha)\\ &\leqslant |I_0|\sum_{k=0}^{\infty}\min\left(1, 2\gamma^k\right)\chi_{[k,k+1)}(\alpha).\end{aligned}$$

Hence, by (2.1),

$$\begin{aligned}\left|\left\{x \in I_0\colon\ \left|f(x) - f_{I_0}\right| > \alpha\right\}\right| &\leqslant \left|\left\{x \in I_0\colon\ \psi(x) > 2\gamma\alpha/\|f\|_*\right\}\right|\\ &\leqslant |I_0|\sum_{k=0}^{\infty}\min\left(2\gamma^k, 1\right)\chi_{[k,k+1)}\left(2\gamma\alpha/\|f\|_*\right).\end{aligned}$$

This estimate holds for any $0<\gamma<1$. Therefore, taking here the infimum over $0<\gamma<1$, we obtain:

$$\left|\left\{x\in I_0:\ \left|f(x)-f_{I_0}\right|>\alpha\right\}\right|\leqslant\varphi\left(\frac{2/\mathrm{e}}{\|f\|_*}\alpha\right)|I_0|,$$

where

$$\varphi(\xi)=\inf_{0<\gamma<1}\sum_{k=0}^{\infty}\min\left(2\gamma^k,1\right)\chi_{[k,k+1)}(\gamma\mathrm{e}\xi).$$

Thus, the theorem would follow from the following estimate:

$$\varphi(\xi)\leqslant\frac{1}{2}\mathrm{e}^{\frac{4}{\mathrm{e}}-\xi}\quad(\xi>0).\tag{2.6}$$

It is easy to see that $\varphi(\xi)=1$ for $0<\xi\leqslant 2/\mathrm{e}$, and in this case (2.6) holds trivially. Next, $\varphi(\xi)=\frac{2}{\mathrm{e}\xi}$ for $2/\mathrm{e}\leqslant\xi\leqslant 4/\mathrm{e}$. Using that the function e^{ξ}/ξ is increasing on $(1,\infty)$ and decreasing on $(0,1)$, we get:

$$\max_{\xi\in[2/\mathrm{e},4/\mathrm{e}]}2\mathrm{e}^{\xi}/\mathrm{e}\xi=\frac{1}{2}\mathrm{e}^{4/\mathrm{e}},$$

verifying (2.6) for $2/\mathrm{e}\leqslant\xi\leqslant 4/\mathrm{e}$.

For $\xi\geqslant 1$ we estimate $\varphi(\xi)$ as follows. Let $\xi\in[m,m+1)$, $m\in\mathbb{N}$. Taking $\gamma_i=i/\mathrm{e}\xi$ for $i=m$ and $i=m+1$, we get:

$$\begin{aligned}\varphi(\xi)&\leqslant 2\min\left(\left(\frac{m}{\mathrm{e}\xi}\right)^m,\left(\frac{m+1}{\mathrm{e}\xi}\right)^{m+1}\right)\\&=2\left(\left(\frac{m}{\mathrm{e}\xi}\right)^m\chi_{[m,\xi_m]}(\xi)+\left(\frac{m+1}{\mathrm{e}\xi}\right)^{m+1}\chi_{[\xi_m,m+1)}(\xi)\right),\end{aligned}\tag{2.7}$$

where $\xi_m=\frac{1}{\mathrm{e}}\frac{(m+1)^{m+1}}{m^m}$. Using the fact that the function e^{ξ}/ξ^m is increasing on (m,∞) and decreasing on $(0,m)$, by (2.7) we obtain that for $\xi\in[m,m+1)$,

$$\varphi(\xi)\mathrm{e}^{\xi}\leqslant 2\left(\frac{m}{\mathrm{e}\xi_m}\right)^m\mathrm{e}^{\xi_m}=2\left(\frac{\mathrm{e}^{\frac{1}{\mathrm{e}}(1+1/m)^m}}{(1+1/m)^m}\right)^{m+1}\equiv c_m.$$

Let us show now that the sequence $\{c_m\}$ is decreasing. This would finish the proof since $c_1=\frac{1}{2}\mathrm{e}^{4/\mathrm{e}}$. Let $\eta(x)=(1+1/x)^x$ for $x>0$, and

$$\nu(x)=\left(\mathrm{e}^{\eta(x)/\mathrm{e}}/\eta(x)\right)^{x+1}.$$

Then $c_m=2\nu(m)$ and hence it suffices to show that $\nu'(x)<0$ for $x\geqslant 1$. We have:

$$\nu'(x)=\nu(x)\left(\log\frac{\mathrm{e}}{\eta(x)}-\left(1-\eta(x)/\mathrm{e}\right)\log(1+1/x)^{1+x}\right).$$

Since $\eta(x)(1+1/x)>\mathrm{e}$, we get $\mu(x)=\frac{\eta(x)}{\mathrm{e}-\eta(x)}>x$. From this and from the fact that the function $(1+1/x)^{1+x}$ is decreasing, we obtain:

$$\left(\mathrm{e}/\eta(x)\right)^{\frac{1}{1-\eta(x)/\mathrm{e}}}=\left(1+1/\mu(x)\right)^{1+\mu(x)}<(1+1/x)^{1+x},$$

which is equivalent to that $\nu'(x)<0$. □

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