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Algebraic Geometry

# On natural deformations of symplectic automorphisms of manifolds of $K3^{[n]}$ type



Déformations naturelles des automorphismes symplectiques sur les variétés de type  $K3^{[n]}$ 

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ARTICLE INFO	ABSTRACT
Article history: Received 25 April 2013 Accepted after revision 24 July 2013 Available online 2 August 2013 Presented by Claire Voisin	In the present paper, we prove that finite symplectic groups of automorphisms of manifolds of $K3^{[n]}$ type can be obtained by deforming natural morphisms arising from K3 surfaces if and only if they satisfy a certain numerical condition. © 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved. R É S U M É
	Dans cette étude, on démontre que tout groupe d'ordre fini des automorphismes symplec- tiques sur les variétés de type $K3^{[n]}$ s'obtient comme déformation des automorphismes naturels provenant d'une surface $K3$ si et seulement si il satisfait une certaine condition numérique. © 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

#### 1. Introduction

The present paper is devoted to a natural question concerning deformations of automorphisms of hyperkähler manifolds. Roughly speaking, given a K3 surface S, the group Aut(S) induces automorphisms of the Hilbert scheme  $S^{[n]}$  of subschemes of length n of S. These automorphisms are called natural. Let X be a hyperkähler manifold deformation equivalent to some  $S^{[n]}$  and let G be a group of automorphisms of X. One can ask whether it is possible to deform X together with G to some  $(S^{[n]}, G)$ , where G is a group of natural automorphisms. In the following we give a positive answer for all finite symplectic automorphism groups whose action on  $H^2(X,\mathbb{Z})$  is the natural one, in the case that (n-1) is primary (cf. Theorem 2.5). We remark that having the natural action on  $H^2(X, \mathbb{Z})$  is a necessary condition, since this action is constant under smooth deformations.

There have been several works concerning automorphisms of K3 surfaces, we will refer to the foundational work of Nikulin [12], later improved by Mukai [11] in the nonabelian case. By the work of Mukai [11], there are 79 possible finite groups of symplectic automorphisms on K3 surfaces and, by a recent classification due to Hashimoto [6], there are 84 different possibilities for their action on  $H^2$ . Our result holds for all these 84 cases as long as the hypothesis of the global Torelli theorem is satisfied.

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In the case of manifolds of  $K3^{[n]}$  type the notion of natural morphisms was introduced by Boissière [4] and further analysed by him and Sarti [5]. In the particular case of symplectic involutions on manifolds of  $K3^{[2]}$  type, our result is proven in [10].

#### 1.1. Notations

If *L* is a lattice and  $G \subset O(L)$ , we denote by  $T_G(L) := L^G$  the *invariant sublattice* and by  $S_G(L) := T_G(L)^{\perp}$  the *coinvariant sublattice*. For  $G \subset Aut(X)$  and  $H^2(X, \mathbb{Z})$  endowed with a quadratic form, we denote  $T_G(X) := T_G(H^2(X, \mathbb{Z}))$  the invariant sublattice and  $S_G(X) := S_G(H^2(X, \mathbb{Z}))$  the coinvariant sublattice. Let *X* be a hyperkähler manifold and let  $G \subset Aut(X)$ . The group *G* is called *symplectic* if it acts trivially on  $H^{2,0}(X)$ , i.e. it preserves the symplectic form. We denote by  $Aut_s(X)$  the subgroup of automorphisms of *X* preserving the symplectic form. We will call manifolds of  $K3^{[n]}$  type all compact Kähler manifolds deformation equivalent to the Hilbert scheme of length *n* subschemes of a K3 surface.

#### 1.2. Preliminaries

In this section we gather some useful results for ease of reference. The reader interested in hyperkähler manifolds can consult [8] and [9] for further references and for a broader treatment of the subject.

A hyperkähler manifold is a simply connected compact Kähler manifold whose  $H^{2,0}$  is generated by a symplectic form.

**Theorem 1.1.** Let X be a hyperkähler manifold of dimension 2n. Then there exists a canonically defined pairing  $(,)_X$  on  $H^2(X, \mathbb{C})$ , the Beauville–Bogomolov pairing, which is a deformation and birational invariant. This form makes  $H^2(X, \mathbb{Z})$  a lattice of signature  $(3, b_2(X) - 3)$ .

For every hyperkähler manifold *X* and every Kähler class  $\omega$ , there exists a family of smooth deformations of *X* over the base  $\mathbb{P}^1 := \{(a, b, c) \in \mathbb{C}^3, a^2 + b^2 + c^2 = 1\}$ . There is a hyperkähler metric *g* associated with  $\omega$ , this metric admits a  $\mathbb{P}^1$  of different complex structures whose classes are generated by  $\omega$ ,  $\sigma + \overline{\sigma}$  and  $\sigma - \overline{\sigma}$ . Fibres of the family are the real manifold (*X*, *g*) with Kähler structure given by the class  $a\omega + b(\sigma + \overline{\sigma}) + c(\sigma - \overline{\sigma})$ . This family is called *twistor family* and denoted by  $TW_{\omega}(X)$ .

**Example 1.** Let X be a hyperkähler manifold of  $K3^{[n]}$  type. Then  $H^2(X, \mathbb{Z})$  endowed with its Beauville–Bogomolov pairing is isomorphic to the lattice:

$$L_n := H^2(K3, \mathbb{Z}) \oplus (2-2n). \tag{1}$$

(2)

If X is hyperkähler, we call a *marking* of X any isometry between  $H^2(X, \mathbb{Z})$  and a lattice M. There exists a moduli space of marked hyperkähler manifolds with  $H^2(X, \mathbb{Z}) \cong M$  and we denote it by  $\mathcal{M}_M$ .

We will often consider the induced action of Aut(X) on  $O(H^2(X, \mathbb{Z}))$  for a manifold X of  $K3^{[n]}$  type. For a general hyperkähler manifold this map might not be injective but in our case it is:

**Lemma 1.2.** Let X be a manifold of  $K3^{[n]}$  type. Then the map

$$\nu(X)$$
: Aut $(X) \to O(H^2(X,\mathbb{Z}))$ 

is injective.

**Proof.** By [7, Theorem 2.1] the kernel of  $\nu(X)$  is invariant under smooth deformations. Beauville [1, Lemma 3] proved that, if *S* is a *K*3 surface with no nontrivial automorphisms, then Aut( $S^{[n]}$ ) = {Id}, therefore {Id} = Ker( $\nu(S^{[n]})$ ) = Ker( $\nu(X)$ ).  $\Box$ 

The following is a very important theorem, which is essential in the proof of our main result. The only truly restrictive hypothesis of Theorem 2.5 is one of the hypotheses of the following:

**Theorem 1.3** (Global Torelli, Huybrechts, Markman and Verbitsky). (See [9, Theorem 6.4].) Let X and Y be two hyperkähler manifolds of  $K3^{[n]}$  type and let n - 1 be a prime power. Suppose  $\psi : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$  is an isometry preserving the Hodge structure. Then there exists a birational map  $\phi : X \to Y$ .

Let *M* be a lattice of signature (3, r). We define  $\Omega_M = \mathbb{P}(\{x \in M \otimes \mathbb{C} \mid x^2 = 0, (x, \bar{x}) > 0\})$  as the *period domain* for the lattice *M*. It is an open subset of a quadric hypersurface inside  $\mathbb{P}(M \otimes \mathbb{C})$ , as proven in [2, Section 8].

In the particular case where  $M \cong H^2(X, \mathbb{Z})$  for some hyperkähler manifold X, there exists a natural map, the *period* map  $\mathcal{P}$ , between the moduli space  $\mathcal{M}_M$  and the period domain  $\Omega_M$ .

Moreover, when Theorem 1.3 holds, two marked manifolds having the same period are birational.

The images of twistor families in  $\mathcal{M}_M$  through the period map are called *twistor lines*. A fundamental property of period domains is that they are connected by twistor lines (see [9, Proposition 3.7] or [3]).

### 2. Deformations of pairs

**Definition 2.1.** Let X be a manifold and let  $G \subset Aut(X)$ . A *G*-deformation of X (or a deformation of the pair (X, G)) consists of the following data:

- a flat family  $\mathcal{X} \xrightarrow{\alpha} B$ , with B connected and smooth  $\alpha$ , and a distinguished point  $0 \in B$  such that  $\mathcal{X}_0 \cong X$ ,
- a faithful action of the group G on  $\mathcal{X}$  inducing fibrewise faithful actions of G.

Two pairs (X, G) and (Y, G) are deformation equivalent if (Y, G) lies in a G-deformation of X.

The first interesting remark is that, to some extent, all finite symplectic automorphism groups of a hyperkähler manifold can be deformed.

**Remark 1.** Let *X* be a hyperkähler manifold such that  $G \subset \operatorname{Aut}_{S}(X)$  and  $|G| < \infty$ . Let  $\omega$  be a *G*-invariant Kähler class. Then  $TW_{\omega}(X)$  is a *G*-deformation of *X* over  $\mathbb{P}^{1}$ .

Let X be a manifold of  $K3^{[n]}$  type and let  $G \subset Aut_s(X)$ . Then there exists a universal local G-deformation of X sitting inside Def(X). It is locally given by the G-invariant part of  $H^1(T_X)$  and it is of dimension  $rank(T_G(X)) - 2$ . For a proof of its existence, we refer to [10].

**Lemma 2.2.** Let *X* and *Y* be two birational manifolds with isomorphic actions of *G* on cohomology, then they have intersecting local *G*-deformations.

**Proof.** Let *X* be birational to *Y* and let the action of *G* on  $H^2(X)$  coincide with the action of *G* on  $H^2(Y)$  induced by the birational transformation between *X* and *Y*. Let us take a representative *U* of Def(*X*) and let *x* be a very general point inside  $U^G$ , which is a representative of the local *G*-deformations of *X* and *Y*. Let  $\mathcal{X}$  and  $\mathcal{Y}$  be the total spaces of these deformations. Let  $\mathcal{Y}_x$  and  $\mathcal{X}_x$  be the two hyperkähler manifolds corresponding to *x* on  $U^G$ . We have  $\text{Pic}(\mathcal{Y}_x) = \text{Pic}(\mathcal{X}_x) = S_G(X)$  and  $\mathcal{Y}_x$  is birational to  $\mathcal{X}_x$ . However any *G*-invariant Kähler class on  $\mathcal{Y}_x$  is orthogonal to  $\text{Pic}(\mathcal{Y}_x)$  and therefore also to the set of effective curves on  $\mathcal{Y}_x$ , which is therefore empty. Thus the Kähler cone of  $\mathcal{Y}_x$  coincides with the positive cone and  $\mathcal{Y}_x = \mathcal{X}_x$ .  $\Box$ 

We remark that the local G-deformations around two birational manifolds might not meet for a nonsymplectic group G.

**Definition 2.3.** Let *S* be a *K*3 surface and let  $G \subset Aut_S(S)$  be a group of symplectic automorphisms on *S*. *G* induces a group of symplectic automorphisms on  $S^{[n]}$  which we still denote as *G*. We call such a pair ( $S^{[n]}$ , *G*) a *natural pair*, following [4]. We call *standard* any pair (*X*, *G*) deformation equivalent to a natural pair.

A natural question is asking under which condition a pair (X, G) is standard. In the rest of the paper we make the following assumption and we prove that it is equivalent to (X, G) being standard in the case that (n - 1) is primary.

**Definition 2.4.** Let X be a manifold of  $K3^{[n]}$  type and let  $G \subset Aut_s(X)$ . The pair (X, G) is *numerically standard* if the following holds:

 $\begin{array}{l} - S_G(X) \cong S_H(S), \\ - T_G(X) \cong T_H(S) \oplus \langle t \rangle, \\ - t^2 = -2(n-1), \ (t, H^2(X, \mathbb{Z})) = 2(n-1)\mathbb{Z}. \end{array}$ 

For some K3 surface S and some  $H \subset Aut_s(S)$  such that  $H \cong G$ .

Notice that for a standard pair (X, G), the group G is numerically standard, since by [5] a natural pair is numerically standard. Now the main result of the paper can be explicitly stated:

**Theorem 2.5.** Let X be a manifold of  $K3^{[n]}$  type and let n - 1 be a prime power. Let  $G \subset Aut_s(X)$  be a finite group of numerically standard automorphisms. Then (X, G) is a standard pair.

**Definition 2.6.** Let *M* be a lattice of signature (3, r) and let  $G \subset O(M)$ . We call  $\Omega_{G,M}$  the set of points  $\sigma$  in the period domain  $\Omega_M$  such that  $\sigma \in T_G(M) \otimes \mathbb{C}$ .

**Definition 2.7.** Let  $\mathcal{M}_n := \mathcal{M}_{L_n}$  be the moduli space of marked manifolds of  $K3^{[n]}$  type and let  $G \subset \operatorname{Aut}_S(X)$  for some marked  $(X, f) \in \mathcal{M}_n$ . Let us denote by G the group of isometries induced by G on the lattice  $L_n$  (well defined by Lemma 1.2) and let  $\Omega_{G,n} := \Omega_{G,L_n}$  be as above. Then we define  $\mathcal{M}_{G,n} \subset \mathcal{M}_n$  as the preimage through the period map of  $\Omega_{G,n}$ .

This definition makes sense for symplectic automorphisms, otherwise periods would not be preserved by the group action. By the following remark the set  $\mathcal{M}_{G,n}$  is the set of marked pairs (X, f) such that  $f^{-1}(S_G(L_n)) \subset \operatorname{Pic}(X)$  for an appropriate marking f and  $\Omega_{G,n}$  is just the period domain  $\Omega_{T_G(L_n)}$ .

**Remark 2.** Let *X* be a hyperkähler manifold and let  $G \subset Aut_s(X)$  be a finite group. Then  $T_G(X)$  contains the transcendental lattice T(X) and  $S_G(X) \subset Pic(X)$ . Moreover  $T_G(X)$  has signature (3, r) for some  $r \ge 0$ . A proof of this fact can be found in [1, Proposition 6].

Therefore we have a *G*-deformation given by a twistor family for every Kähler class contained in  $T_G(X) \otimes \mathbb{R}$ . If *X* is very general, it has  $T(X) = T_G(X)$ . Moreover, its Kähler cone is one connected component of the set of all positive 1, 1 classes in  $T_G(X) \otimes \mathbb{R}$ . Twistor lines given by these classes are precisely all twistor lines through the period of *X*.

**Proof of Theorem 2.5.** Let *X* be a manifold of  $K3^{[n]}$  type and let n - 1 be a prime power. Let  $G \subset Aut_s(X)$  be a finite numerically standard group of symplectic automorphisms. Since  $\Omega_{G,n}$  is connected by twistor lines, (X, G) is deformation equivalent to (Y, G) and  $\mathcal{P}(Y, f) = \mathcal{P}(S^{[n]}, f') \in \Omega_{G,n}$ . Here *S* is a *K*3 surface with  $G \subset Aut_s(S)$  and  $Pi(Y, f) = \mathcal{P}(S^{[n]}, f') \in \Omega_{G,n}$ . Here *S* is a birational map  $\phi$  between *Y* and  $S^{[n]}$ , which gives an induced action of *G* on  $S^{[n]}$  (possibly nonregular). Let us denote by *H* the group induced on  $S^{[n]}$  by  $\phi$  and let us keep calling *G* the group induced by the automorphisms of *S*. We obtain our claim by proving that H = G (as actions on  $S^{[n]}$ ), since in that case (Y, G) and  $(S^{[n]}, H)$  would be deformation equivalent through their local universal *G*-deformations by Lemma 2.2.

Notice that, by the assumption on the numerical standardness, the actions of *G* and *H* already coincide on  $H^2(S^{[n]}, \mathbb{Z})$ . Let now  $g \in G$  and let *h* be the element of *H* such that  $g^* = h^*$  in  $H^2(S^{[n]}, \mathbb{Z})$ . Let *r* be the order of *g*. Then  $g \circ h^{r-1}$  induces the identity on  $H^2(S^{[n]}, \mathbb{Z})$ . Therefore, by Lemma 1.2,  $g^{-1} = h^{r-1}$ , which implies G = H as a group of automorphisms of  $S^{[n]}$ .  $\Box$ 

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