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Combinatorics

The indecomposable tournaments T with $|W_5(T)| = |T| - 2$

Les tournois indécomposables T tels que $|W_5(T)| = |T| - 2$

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A R T I C L E   I N F O

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A B S T R A C T

We consider a tournament $T = (V, A)$. For $X \subseteq V$, the subtournament of T induced by X is $T[X] = (X, A \cap (X \times X))$. An interval of T is a subset X of V such that, for $a, b \in X$ and $x \in V \setminus X$, $(a, x) \in A$ if and only if $(b, x) \in A$. The trivial intervals of T are $\emptyset$, $\{x\}$ $(x \in V)$ and V. A tournament is indecomposable if all its intervals are trivial. For $n \geqslant 2$, W_{2n+1} denotes the unique indecomposable tournament defined on $\{0, \ldots, 2n\}$ such that $W_{2n+1}[\{0, \ldots, 2n-1\}]$ is the usual total order. Given an indecomposable tournament T, $W_5(T)$ denotes the set of $v \in V$ such that there is $W \subseteq V$ satisfying $v \in W$ and $T[W]$ is isomorphic to W_5. Latka [6] characterized the indecomposable tournaments T such that $W_5(T) = \emptyset$. The authors [1] proved that if $W_5(T) \neq \emptyset$, then $|W_5(T)| \geqslant |V| - 2$. In this note, we characterize the indecomposable tournaments T such that $|W_5(T)| = |V| - 2$.

R É S U M É

Considérons un tournoi $T = (V, A)$. Pour $X \subseteq V$, le sous-tournoi de T induit par X est $T[X] = (X, A \cap (X \times X))$. Un intervalle de T est une partie X de V telle que, pour tous $a, b \in X$ et $x \in V \setminus X$, $(a, x) \in A$ si et seulement si $(b, x) \in A$. Les intervalles triviaux de T sont $\emptyset$, $\{x\}$ $(x \in V)$ et V. Un tournoi est indécomposable si tous ses intervalles sont triviaux. Pour $n \geqslant 2$, W_{2n+1} est l'unique tournoi indécomposable défini sur $\{0, \ldots, 2n\}$ tel que $W_{2n+1}[\{0, \ldots, 2n-1\}]$ est l'ordre total usuel. Étant donné un tournoi indécomposable T, $W_5(T)$ désigne l'ensemble des sommets $v \in V$ pour lesquels il existe une partie W de V telle que $v \in W$ et $T[W]$ est isomorphe à W_5. Latka [6] a caractérisé les tournois indécomposables T tels que $W_5(T) = \emptyset$. Les auteurs [1] ont prouvé que, si $W_5(T) \neq \emptyset$, alors $|W_5(T)| \geqslant |V| - 2$. Dans cette note, nous caractérisons les tournois indécomposables T tels que $|W_5(T)| = |V| - 2$.

1. Introduction

A *tournament* $T = (V(T), A(T))$ (or (V, A)) consists of a finite set V of *vertices* together with a set A of ordered pairs of distinct vertices, called *arcs*, such that, for all $x \neq y \in V$, $(x, y) \in A$ if and only if $(y, x) \notin A$. The *cardinality* of T, denoted by $|T|$, is that of $V(T)$. Given a tournament $T = (V, A)$, with each subset X of V is associated the *subtournament* $T[X] = (X, A \cap (X \times X))$ of T induced by X. For $x \in V$, the subtournament $T[V \setminus \{x\}]$ is denoted by $T - x$. Two tournaments

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$T = (V, A)$ and $T' = (V', A')$ are *isomorphic*, which is denoted by $T \simeq T'$, if there exists an *isomorphism* from T onto T', i.e., a bijection f from V onto V' such that for all $x, y \in V$, $(x, y) \in A$ if and only if $(f(x), f(y)) \in A'$. We say that a tournament T' *embeds* into a tournament T if T' is isomorphic to a subtournament of T. Otherwise, we say that T *omits* T'. A tournament is said to be *transitive* if it omits the tournament $C_3 = (\{0, 1, 2\}, \{(0, 1), (1, 2), (2, 0)\})$. For a finite subset V of $\mathbb{N}$, we denote by $\overrightarrow{V}$ the usual *total order* defined on V, i.e., the transitive tournament $(V, \{(i, j)\colon i < j\})$.

Let $T = (V, A)$ be a tournament. For two vertices $x \neq y \in V$, the notation $x \to y$ signifies that $(x, y) \in A$. Similarly, given $x \in V$ and $Y \subseteq V$, the notation $x \to Y$ (resp. $Y \to x$) means that $x \to y$ (resp. $y \to x$) for all $y \in Y$. Given $x \in V$, we set $N^+_T(x) = \{y \in V\colon x \to y\}$. A subset I of V is an *interval* [4,5,8] of T provided that, for all $x \in V \setminus I$, $x \to I$ or $I \to x$. For example, $\emptyset$, $\{x\}$, where $x \in V$, and V are intervals of T, called *trivial* intervals. A tournament is *indecomposable* [5,8] if all its intervals are trivial; otherwise it is *decomposable*. Notice that a tournament $T = (V, A)$ and its *dual* $T^\star = (V, \{(x, y)\colon (y, x) \in A\})$ share the same intervals. Hence, T is indecomposable if and only if $T^\star$ is. For all $n \in \mathbb{N} \setminus \{0\}$, the set $\{0, \ldots, n-1\}$ is denoted by $\mathbb{N}_n$.

For $n \geqslant 2$, we introduce the tournament W_{2n+1} defined on $\mathbb{N}_{2n+1}$ as follows: $W_{2n+1}[\mathbb{N}_{2n}] = \overrightarrow{\mathbb{N}_{2n}}$ and $N^+_{W_{2n+1}}(2n) = \{2i\colon i \in \mathbb{N}_n\}$. In 2003, B.J. Latka [6] characterized the indecomposable tournaments omitting the tournament W_5. In order to present this characterization, we introduce the tournaments T_{2n+1}, U_{2n+1} defined on $\mathbb{N}_{2n+1}$, where $n \geqslant 2$, and the *Paley* tournament P_7 defined on $\mathbb{N}_7$ as follows.

- $A(T_{2n+1}) = \{(i, j)\colon j - i \in \{1, \ldots, n\} \bmod 2n+1\}$.
- $A(T_{2n+1}) \setminus A(U_{2n+1}) = A(T_{2n+1}[\{n+1, \ldots, 2n\}])$.
- $A(P_7) = \{(i, j)\colon j - i \in \{1, 2, 4\} \bmod 7\}$.

Notice that for all $x \neq y \in \mathbb{N}_7$, $P_7 - x \simeq P_7 - y$, and let $B_6 = P_7 - 6$.

Theorem 1.1. *(See [6].) Up to isomorphism, the indecomposable tournaments on at least* 5 *vertices and omitting* W_5 *are the tournaments* B_6, P_7, T_{2n+1} *and* U_{2n+1}, *where* $n \geqslant 2$.

In 2012, the authors were interested in the set $W_5(T)$ of the vertices x of an indecomposable tournament $T = (V, A)$ for which there exists a subset X of V such that $x \in X$ and $T[X] \simeq W_5$. They obtained the following.

Theorem 1.2. *(See [1].) Let* T *be an indecomposable tournament into which* W_5 *embeds. Then,* $|W_5(T)| \geqslant |T| - 2$. *If, in addition,* $|T|$ *is even, then* $|W_5(T)| \geqslant |T| - 1$.

In this note, we characterize the class $\mathscr{T}$ of the indecomposable tournaments T on at least 3 vertices such that $|W_5(T)| = |T| - 2$. This answers [1, Problem 4.4].

2. Partially critical tournaments and the class $\mathscr{T}$

Our characterization of the tournaments of the class $\mathscr{T}$ requires the notion of partial criticality. This notion is defined in terms of critical vertices. A vertex x of an indecomposable tournament T is *critical* [8] if $T - x$ is decomposable. The set of non-critical vertices of an indecomposable tournament T was introduced in [7]. It is called the *support* of T and is denoted by $\sigma(T)$. Let T be an indecomposable tournament and let X be a subset of $V(T)$ such that $|X| \geqslant 3$; we say that T is *partially critical according to* $T[X]$ (or $T[X]$*-critical*) [3] if $T[X]$ is indecomposable and if $\sigma(T) \subseteq X$.

The result below point out the partial criticality structure of the tournaments of the class $\mathscr{T}$.

Proposition 2.1. *Let* $T = (V, A)$ *be a tournament of the class* $\mathscr{T}$. *Then, we have* $V \setminus W_5(T) = \sigma(T)$. *Moreover, there exists* $z \in W_5(T)$ *such that* $T[(V \setminus W_5(T)) \cup \{z\}] \simeq C_3$. *In particular,* T *is* $T[(V \setminus W_5(T)) \cup \{z\}]$*-critical.*

Proposition 2.1 leads us to consider the characterization of partially critical tournaments as a basic tool in our construction of the tournaments of the class $\mathscr{T}$. Partially critical tournaments were characterized in [7]. In order to recall this characterization, we need some additional notations. Given a tournament $T = (V, A)$, with each subset X of V, such that $|X| \geqslant 3$ and $T[X]$ is indecomposable, are associated the following subsets of $V \setminus X$.

- $X^- = \{x \in V \setminus X\colon x \to X\}$ and $X^+ = \{x \in V \setminus X\colon X \to x\}$.
- For all $u \in X$, $X^-(u) = \{x \in V \setminus X\colon \{u, x\}$ is an interval of $T[X \cup \{x\}]$ and $x \to u\}$ and $X^+(u) = \{x \in V \setminus X\colon \{u, x\}$ is an interval of $T[X \cup \{x\}]$ and $u \to x\}$.
- $\mathrm{Ext}(X) = \{x \in V \setminus X\colon T[X \cup \{x\}]$ is indecomposable$\}$.

The family $\{\mathrm{Ext}(X), X^-, X^+\} \cup \{X^-(u)\colon u \in X\} \cup \{X^+(u)\colon u \in X\}$ is denoted by q^T_X.

A *graph* $G = (V(G), E(G))$ (or (V, E)) consists of a finite set V of vertices together with a set E of unordered pairs of distinct vertices. Given a vertex x of a graph $G = (V, E)$, we set $N_G(x) = \{y \in V, \{x, y\} \in E\}$. With each subset X of V is associated the *subgraph* $G[X] = (X, E \cap \binom{X}{2})$ of G induced by X. An isomorphism from a graph $G = (V, E)$ onto a graph $G' = (V', E')$ is a bijection f from V onto V' such that, for all $x, y \in V$, $\{x, y\} \in E$ if and only if $\{f(x), f(y)\} \in E'$. A graph G is *connected* if for all $x \neq y \in V(G)$, there is a sequence $x_0 = x, \ldots, x_m = y$ of vertices of G such that for all $i \in \mathbb{N}_m$, $\{x_i, x_{i+1}\} \in E(G)$. For example, given $n \geqslant 1$, the graph $G_{2n} = (\mathbb{N}_{2n}, \{\{x, y\} \in \binom{\mathbb{N}_{2n}}{2}: |y - x| \geqslant n\})$ is connected. A *connected component* of a graph G is a maximal subset X of $V(G)$ (with respect to inclusion) such that $G[X]$ is connected. The set of the connected components of G is a partition of $V(G)$, denoted by $\mathscr{C}(G)$. Let $T = (V, A)$ be an indecomposable tournament. With each subset X of V such that $|X| \geqslant 3$ and $T[X]$ is indecomposable, is associated its *outside* graph G^T_X defined by $V(G^T_X) = V \setminus X$ and $E(G^T_X) = \{\{x, y\} \in \binom{V \setminus X}{2}: T[X \cup \{x, y\}]$ is indecomposable$\}$. We now present the characterization of the partially critical tournaments.

Theorem 2.2. *(See [7].) Consider a tournament $T = (V, A)$ with a subset X of V such that $|X| \geqslant 3$ and $T[X]$ is indecomposable. The tournament T is $T[X]$-critical if and only if the assertions below hold.*

(i) $\mathrm{Ext}(X) = \emptyset$.
(ii) *For all $u \in X$, the tournaments $T[X^-(u) \cup X^+(u) \cup \{u\}]$ and $T[X^- \cup X^+ \cup \{u\}]$ are transitive.*
(iii) *For each $Q \in \mathscr{C}(G^T_X)$, there is an isomorphism f from G_{2n} onto $G^T_X[Q]$ such that $Q_1, Q_2 \in q^T_X$, where $Q_1 = f(\mathbb{N}_n)$ and $Q_2 = f(\mathbb{N}_{2n} \setminus \mathbb{N}_n)$. Moreover, for all $x \in Q_i$ ($i = 1$ or 2), $|N_{G^T_X}(x)| = |N^+_{T[Q_i]}(x)| + 1$ (resp. $n - |N^+_{T[Q_i]}(x)|$) if $Q_i = X^+$ or $X^-(u)$ (resp. $Q_i = X^-$ or $X^+(u)$), where $u \in X$.*

The next corollary follows from Theorem 2.2.

Corollary 2.3. *Let T be a $T[X]$-critical tournament, T is entirely determined up to isomorphism by giving $T[X]$, q^T_X and $\mathscr{C}(G^T_X)$. Moreover, the tournament T is exactly determined by giving, in addition, either the graphs $G^T_X[Q]$ for any $Q \in \mathscr{C}(G^T_X)$, or the transitive tournaments $T[Y]$ for any $Y \in q^T_X$.*

We underline the importance of Theorem 2.2 and Corollary 2.3 in our description of the tournaments of the class $\mathscr{T}$. Indeed, these tournaments are introduced up to isomorphism as C_3-critical tournaments T defined by giving $\mathscr{C}(G^T_{\mathbb{N}_3})$ in terms of the nonempty elements of $q^T_{\mathbb{N}_3}$. Fig. 1 illustrates a tournament obtained from such information. We refer to [7, Discussion] for more details about this purpose.

We now introduce the class $\mathscr{H}$ (resp. $\mathscr{I}$, $\mathscr{J}$, $\mathscr{K}$, $\mathscr{L}$) of the C_3-critical tournaments H (resp. I, J, K, L) such that:

- $\mathscr{C}(G^H_{\mathbb{N}_3}) = \{\mathbb{N}^+_3(0) \cup \mathbb{N}^-_3, \mathbb{N}^+_3 \cup \mathbb{N}^-_3(1)\}$ (see Fig. 1);
- $\mathscr{C}(G^I_{\mathbb{N}_3}) = \{\mathbb{N}^+_3(0) \cup \mathbb{N}^+_3(2), \mathbb{N}^+_3(1) \cup \mathbb{N}^-_3(0)\}$;
- $\mathscr{C}(G^J_{\mathbb{N}_3}) = \{\mathbb{N}^+_3(1) \cup \mathbb{N}^-_3, \mathbb{N}^-_3(1) \cup \mathbb{N}^-_3(0)\}$;
- $\mathscr{C}(G^K_{\mathbb{N}_3}) = \{\mathbb{N}^+_3(1) \cup \mathbb{N}^-_3, \mathbb{N}^+_3(0) \cup \mathbb{N}^-_3(2)\}$;
- $\mathscr{C}(G^L_{\mathbb{N}_3}) = \{\mathbb{N}^+_3(1) \cup \mathbb{N}^-_3, \mathbb{N}^+_3(0) \cup \mathbb{N}^-_3(2), \mathbb{N}^+_3 \cup \mathbb{N}^-_3(0)\}$.

Notice that for $\mathscr{X} = \mathscr{H}$, $\mathscr{I}$, $\mathscr{J}$ or $\mathscr{K}$, $\{|V(T)|: T \in \mathscr{X}\} = \{2n+1: n \geqslant 3\}$ and $\{|V(T)|: T \in \mathscr{L}\} = \{2n+1: n \geqslant 4\}$. We denote by $\mathscr{H}^\star$ (resp. $\mathscr{I}^\star$, $\mathscr{J}^\star$, $\mathscr{K}^\star$, $\mathscr{L}^\star$) the class of the tournaments $T^\star$, where $T \in \mathscr{H}$ (resp. $\mathscr{I}$, $\mathscr{J}$, $\mathscr{K}$, $\mathscr{L}$).

Remark 1. We have $\mathscr{H}^\star = \mathscr{H}$ and $\mathscr{I}^\star = \mathscr{I}$.

By setting $\mathscr{M} = \mathscr{H} \cup \mathscr{I} \cup \mathscr{J} \cup \mathscr{J}^\star \cup \mathscr{K} \cup \mathscr{K}^\star \cup \mathscr{L} \cup \mathscr{L}^\star$, we state our main result as follows.

Theorem 2.4. *Up to isomorphism, the tournaments of the class $\mathscr{T}$ are those of the class $\mathscr{M}$. Moreover, for all $T \in \mathscr{M}$, we have $V(T) \setminus W_5(T) = \sigma(T) = \{0, 1\}$.*

3. The sketch of the proof of Theorem 2.4

The complete proof of Theorem 2.4 can be found in [2], we give here the main ideas. Proposition 2.1 leads us to partition the tournaments T of the class $\mathscr{T}$ according to the different values of an invariant $c(T)$ defined as follows. For $T \in \mathscr{T}$, $c(T)$ is the minimum of $|\mathscr{C}(G^T_{\sigma(T) \cup \{x\}})|$, the minimum being taken over all the vertices x of $W_5(T)$ such that $T[\sigma(T) \cup \{x\}] \simeq C_3$. Notice that $c(T) = c(T^\star)$. As T is $T[\sigma(T) \cup \{x\}]$-critical by Proposition 2.1, then $c(T) \leqslant 4$. Moreover, $c(T) \geqslant 2$ by the following lemma.

Lemma 3.1. *Given a C_3-critical tournament T on at least 5 vertices, if $G^T_{\mathbb{N}_3}$ is connected, then $\sigma(T) = \emptyset$.*

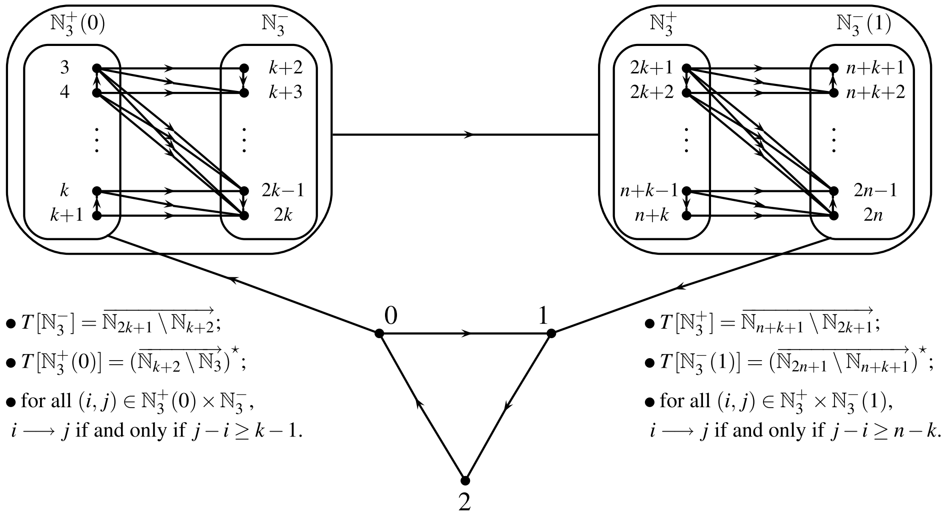

Fig. 1. A tournament T of the class $\mathscr{H}$.

Theorem 2.4 results from the following propositions.

Proposition 3.2. *For all tournament T of the class $\mathscr{M}$, we have $V(T) \setminus W_5(T) = \sigma(T) = \{0, 1\}$.*

Proposition 3.3. *Up to isomorphism, the tournaments T of the class $\mathscr{T}$ such that $c(T) = 2$ are those of the class $\mathscr{M} \setminus (\mathscr{L} \cup \mathscr{L}^\star)$.*

Proposition 3.4. *Up to isomorphism, the tournaments T of the class $\mathscr{T}$ such that $c(T) = 3$ are those of the class $\mathscr{L} \cup \mathscr{L}^\star$.*

Proposition 3.5. *For any tournament T of the class $\mathscr{T}$, we have $c(T) = 2$ or 3.*

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