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Partial differential equations

Effective stability for slow time-dependent near-integrable Hamiltonians and application

Stabilité effective pour des hamiltoniens presque intégrables lentement non autonomes et application

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ABSTRACT

The aim of this note is to prove a result of effective stability for a non-autonomous perturbation of an integrable Hamiltonian system, provided that the perturbation depends slowly on time. Then we use this result to clarify and extend a stability result of Giorgilli and Zehnder for a mechanical system with an arbitrary time-dependent potential.

RÉSUMÉ

Le but de cette note est de démontrer un résultat de stabilité effective pour une perturbation non autonome d'un système hamiltonien intégrable, sous la condition que la perturbation dépende lentement du temps. Nous utilisons ensuite ce résultat pour clarifier et généraliser un résultat de stabilité de Giorgilli et Zehnder pour des systèmes mécaniques dont le potentiel dépend arbitrairement du temps.

1. Introduction

Let $n \in \mathbb{N}$, $n \geqslant 2$, $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and consider the Hamiltonian system defined by $H : \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$,

$$H(\theta, I, t) = h(I) + \varepsilon f(\theta, I, t), \quad (\theta, I, t) = (\theta_1, \dots, \theta_n, I_1, \dots, I_n, t) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R},\ \varepsilon > 0. \tag{1}$$

Nekhoroshev proved [11] that, whenever h is steep (see Section 2 for a definition), $f(\theta, I, t) = f(\theta, I)$ is time-independent and H is real-analytic, there exist positive constants $\varepsilon_0, c_1, c_2, c_3, a, b$ such that for all $\varepsilon \leqslant \varepsilon_0$ and all solutions $(\theta(t), I(t))$, if $|t| \leqslant c_2 \exp(c_3 \varepsilon^{-a})$, then we have the following stability estimate:

$$\left|I(t) - I(0)\right| = \max_{1 \leqslant i \leqslant n} \left|I_i(t) - I_i(0)\right| \leqslant c_1 \varepsilon^b. \tag{2}$$

In the particular case where h is (strictly uniformly) convex or quasi-convex, following a work of Lochak [7] it was proved [9,13], using preservation of energy arguments, that one can choose $a = b = (2n)^{-1}$ in (2), and that these values are close to optimal (in the general steep case, however, there are still no realistic values for these stability exponents a and b).

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The purpose of this note is to discuss to which extent a stability estimate similar to (2) holds true if the perturbation is allowed to depend on time.

Assume first that f depends periodically on time, that is $f(\theta, I, t) = f(\theta, I, t + T)$ in (1) for some $T > 0$ (we may assume $T = 1$ by a time scaling). Removing the time dependence by adding an extra degree of freedom, the Hamiltonian is equivalent to:

$$\tilde{H}(\theta, \varphi, I, J) = \tilde{h}(I, J) + \varepsilon f(\theta, \varphi, I), \quad (\theta, \varphi = t, I, J) \in \mathbb{T}^n \times \mathbb{T} \times \mathbb{R}^n \times \mathbb{R}, \ \tilde{h}(I, J) = h(I) + J.$$

It turns out that if h is convex, then $\tilde{h}$ is quasi-convex and so (2) holds true with $a = b = (2(n+1))^{-1}$. In general, it is possible for $\tilde{h}$ to be steep, in which case (2) is satisfied, but it is not clear how to formulate a condition on h (and not on $\tilde{h}$) to ensure that (2) holds true.

Now assume that f depends quasi-periodically on time, that is $f(\theta, I, t) = F(\theta, I, t\omega)$ in (1) for some function $F : \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T}^m \to \mathbb{R}$ and some vector $\omega \in \mathbb{R}^m$, which we can assume to be non-resonant ($k \cdot \omega \neq 0$ for any non-zero $k \in \mathbb{Z}^m$). As before, the time dependence can be removed by adding m degrees of freedom and we are led to consider $\tilde{H}(\theta, \varphi, I, J) = \tilde{h}(I, J) + \varepsilon f(\theta, \varphi, I)$, but this time:

$$(\theta, \varphi = t\omega, I, J) \in \mathbb{T}^n \times \mathbb{T}^m \times \mathbb{R}^n \times \mathbb{R}^m, \qquad \tilde{h}(I, J) = h(I) + \omega \cdot J.$$

It was conjectured by Chirikov [3], and then again by Lochak [8], that if h is convex and ω satisfies a Diophantine condition of exponent $\tau \geqslant m - 1$ (there exists a constant $\gamma > 0$ such that $|k \cdot \omega| \geqslant \gamma |k|^{-\tau}$ for any non-zero $k \in \mathbb{Z}^m$), then the estimate (2) holds true and, moreover, we can choose $a = b = (2(n+1+\tau))^{-1}$. If $m = 1$, then $\tau = 0$ and we are in the periodic case, so the conjecture is true. However, if $m > 1$, $\tilde{h}$ cannot be steep and the problem is still completely open. Even though the conjecture is sometimes considered as granted (without the explicit values for a and b, see, for instance, [6]), there is still no proof. Needless to say that the situation in the general case (without the convexity assumption on h) is even more complicated.

In a different direction, Giorgilli and Zehnder [4] considered the following time-dependent Hamiltonian:

$$G(\theta, I, t) = h_2(I) + V(\theta, t), \quad (\theta, I, t) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}, \ h_2(I) = I_1^2 + \cdots + I_n^2,$$

and proved the following Nekhoroshev-type result: if G is real-analytic and V is uniformly bounded, then for R sufficiently large, if I_0 belongs to the open ball B_R of radius R centered at the origin, then $I(t) \in B_{2R}$ for $|t| \leqslant c_2 \exp(c_3 R^d)$ for some positive constants c_2, c_3, and d. Even though such a system is clearly not of the form (1), the fact that no restriction on the time dependence is imposed in their result has led to several confusions. In [4], the authors themselves assert that "extra work is needed because the time dependence is not assumed to be periodic or quasi-periodic". Even more surprising, one can read (in [10] for instance) that this result implies that the estimate (2) holds true for (1) without any restriction on the time dependence. Concerning the latter assertion, it is simply wrong and it seems very unlikely to have a non-trivial stability estimate for (1) with an arbitrary time dependence. As for the former assertion, it is not difficult to see that the system considered in [4] can be given the form (1), but with a perturbation depending "slowly" (and not arbitrarily) on time (see Section 2 for a definition of what we mean by "slowly" depending on time, and Section 3 for more details on the system considered in [4]). We will show in Section 2 that for a Hamiltonian system depending "slowly" on time, essentially classical techniques can be used to prove that (2) holds true, and that the non-periodicity or non-quasi-periodicity of time in this restricted context plays absolutely no role (as a matter of fact, we already explained that, for a periodic or quasi-periodic time dependence that is not slow, basic questions are still open). Then, in Section 3, we will use this result to derive, in a simpler way, a more general statement than the one contained in [4].

2. A stability result

For a given $\rho > 0$, recall that B_ρ is the open ball in $\mathbb{R}^n$ of radius ρ (with respect to the supremum norm) around the origin. A function $h \in C^2(B_\rho)$ is said to be steep if, for any affine subspace S of $\mathbb{R}^n$ intersecting B_ρ, the restriction $h_{|S}$ has only isolated critical points (it is not the original definition of Nekhoroshev, but it is equivalent to it, see [5] and [12]). We will assume that the operator norm $|\nabla^2 h(I)|$ is bounded uniformly in $I \in B_\rho$. Then, given $r, s > 0$, let us define the complex domain:

$$\mathcal{D}_{r,s} = \big\{(\theta, I, t) \in (\mathbb{C}^n/\mathbb{Z}^n) \times \mathbb{C}^n \times \mathbb{C} \ \big| \ \big|(\mathrm{Im}(\theta_1), \ldots, \mathrm{Im}(\theta_n))\big| < s, \ \big|\mathrm{Im}(t)\big| < s, \ d(I, B_\rho) < r\big\},$$

where the distance d is induced by the supremum norm. For a fixed constant $\lambda > 0$ and a "small" parameter $0 < \varepsilon \leqslant 1$, we consider $H(\theta, I, t) = h(I) + \varepsilon f(\theta, I, \varepsilon^\lambda t)$ defined on $\mathcal{D}_{r,s}$, real-analytic (that is H is analytic and real-valued for real arguments), and we assume that $|f(\theta, I, t)| \leqslant 1$ for any $(\theta, I, t) \in \mathcal{D}_{r,s}$.

Theorem 2.1. *Under the previous assumptions, there exist positive constants $\varepsilon_0, c_1, c_2, c_3$, that depend on $n, \rho, h, r, s, \lambda$, and positive constants a, b that depend only on n, h, such that if $\varepsilon \leqslant \varepsilon_0$, for all solutions $(\theta(t), I(t))$ of the Hamiltonian system defined by H, if $I(0) \in B_{\rho/2}$, then the estimate $|I(t) - I(0)| \leqslant c_1 \varepsilon^b$ holds true for all time $|t| \leqslant c_2 \exp(c_3 \varepsilon^{-a})$.*

Note that a general time-dependent perturbation of an integrable Hamiltonian system is simply of the form $h(I) + \varepsilon f(\theta, I, t)$; the specific Hamiltonian H that we considered above (with $f(\theta, I, \varepsilon^\lambda t)$ instead of $f(\theta, I, t)$) is what we call a "slow" time-dependent perturbation of an integrable Hamiltonian system (a more general definition of a slow time-dependent perturbation would be to replace $\varepsilon^\lambda t$ by $\mu(\varepsilon)t$ for an increasing analytic function $\mu : (0, +\infty) \to (0, +\infty)$ satisfying $\lim_{\varepsilon\to 0} \mu(\varepsilon) = 0$, and a statement similar to Theorem 2.1 holds true in this setting). The case of a general (non-slow) time-dependent perturbation, which is the most interesting and most difficult, was discussed in the Introduction and we expect that no non-trivial stability results can be obtained unless the time dependence is driven by a recurrent dynamics (we will not try to make this precise, but a simple and yet open case is when the time dependence is quasi-periodic). For a slow time-dependent perturbation, Theorem 2.1 claims that we have effective stability exactly as if the perturbation were time independent (in particular, quasi-periodicity or any other assumption is irrelevant in the context of a slow time-dependent perturbation).

Let us now explain the proof of Theorem 2.1, which follows from the arguments exposed in [2] or [1], up to some technical points that we shall detail now. First we remove the time dependence: we let $x = \varepsilon^\lambda t$ and we introduce a variable y canonically conjugated with x, so that the Hamiltonian is equivalent to:

$$\tilde{H}(\theta, I, x, y) = h(I) + \varepsilon^\lambda y + \varepsilon f(\theta, I, x) = h(I) + \tilde{f}(\theta, I, x, y), \quad (\theta, I, x, y) \in \tilde{\mathcal{D}}_{r,s}, \tag{3}$$

where $\tilde{\mathcal{D}}_{r,s} = \mathcal{D}_{r,s} \times \{y \in \mathbb{C} \mid |\mathrm{Im}(y)| < s\}$. The fact that the dependence on time is slow allows us to keep the integrable part fixed when removing the time dependence, as one can consider that the extra degree of freedom only affects the perturbation. The new perturbation $\tilde{f}$ depends on parameters or "degenerate" variables x and y (degenerate since they are not present in the integrable part), and such systems were already considered by Nekhoroshev [11]. However, a difficulty arises: for subsequent arguments, it is important for the (real part of the) variable y to be unbounded, which is indeed the case by our definition of $\tilde{\mathcal{D}}_{r,s}$; but on this extended domain $\tilde{f}$ is unbounded and this implies that $\tilde{H}$ in (3) is not a perturbation of h. Yet the Hamiltonian vector field $X_{\tilde{H}}$ can be considered as a perturbation of X_h, as $X_{\tilde{f}} = (\partial_I \tilde{f}, -\partial_\theta \tilde{f}, \partial_y \tilde{f}, -\partial_x \tilde{f}) = (\varepsilon \partial_I f, -\varepsilon \partial_\theta f, \varepsilon^\lambda, -\varepsilon \partial_t f)$, and so $X_{\tilde{f}}$ is bounded (by a Cauchy estimate) and small on the domain $\tilde{\mathcal{D}}_{r/2,s/2}$, for instance. As a consequence, even when h is convex, one cannot use preservation of energy arguments as it is the case in [9,13,7], and in general one has to use a perturbation theory that deals only with vector fields: the proofs in [2] and [1] accommodate both requirements. Now the analytic part of [2] and [1] goes exactly the same way for (3), by simply considering x and y as "dummy" variables: given an integer parameter $m \geqslant 1$ which will be determined (by the geometric part of the proof) in terms of ε, on suitable domains resonant normal forms with a remainder of size bounded by a constant times $\varepsilon^\lambda \mathrm{e}^{-m}$ are constructed (note that the size of the perturbation $X_{\tilde{f}}$ is of order $\varepsilon^{\min\{1,\lambda\}}$, but its "effective" size is just of order ε, and so m will be determined in terms of ε and not ε^λ; ε^λ just enters the pre-factor in the exponential and it can be proved that it will not affect the radius of confinement with suitable choices of ε_0 and c_3). The geometric part of [2] and [1] also goes exactly the same way since the time of escape (of the domain) of the degenerate variables x and y is infinite (as the domain is unbounded in these directions), m is eventually chosen proportional to ε^{-a}, the radius of confinement is chosen proportional to ε^b and the stability time is proven to be at least a constant times e^{-m}.

Now let us add two remarks on the statement of Theorem 2.1. First, the exponents a and b are the same as in (2) when the perturbation is time independent. It is reasonable to expect that if h is convex, then $a = b = (2n)^{-1}$ in Theorem 2.1, but we already explained that we cannot use preservation of energy arguments and so we cannot reach these values: the problem actually reduces to the problem of finding realistic values of a and b in the general steep case, which is still open. Then, using [2] and [1], the statement of Theorem 2.1 can be generalized in two ways: using [2] the statement holds true for the much wider class of Diophantine steep functions introduced by Niederman (which is a prevalent class of functions), using [1] the statement holds true for α-Gevrey Hamiltonians for $\alpha \geqslant 1$ (with $\exp(c_3\varepsilon^{-a})$ replaced by $\exp(c_3\varepsilon^{-\alpha^{-1}a})$), recall that 1-Gevrey is real-analytic) and for C^k Hamiltonians, $k \geqslant n+1$ (with $\exp(c_3\varepsilon^{-a})$ replaced by $c_3\varepsilon^{-k^* a}$, with k^* the largest integer $l \geqslant 1$ such that $k \geqslant \ln + 1$).

3. An application

Now we come back to the problem studied in [4], and more generally we consider, for an integer $p \geqslant 2$,

$$G(\theta, I, t) = h_p(I) + V(\theta, t), \quad (\theta, I, t) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}, \ h_p(I) = I_1^p + \cdots + I_n^p.$$

The case $p = 2$ corresponds to [4] and h_2 is convex; in general, the function h_p is not convex, but it is steep. The function V is assumed to be real-analytic, defined on $\mathcal{D}_s = \{(\theta, t) \in (\mathbb{C}^n/\mathbb{Z}^n) \times \mathbb{C} \mid |(\mathrm{Im}(\theta_1), \ldots, \mathrm{Im}(\theta_n)| < s, \ |\mathrm{Im}(t)| < s\}$, and it is assumed that $|V(\theta, t)| \leqslant 1$ for all $(\theta, t) \in \mathcal{D}_s$.

Theorem 3.1. *Under the previous assumptions, there exist positive constants R_0, c_1, c_2, c_3 that depend on n, p, s, and positive constants a', b' that depend only on n, p, such that if $R \geqslant R_0$, for all solutions $(\theta(t), I(t))$ of the Hamiltonian system defined by G, if $I(0) \in B_R$, then the estimate $|I(t) - I(0)| \leqslant c_1 R^{1-b'}$ holds true for all time $|t| \leqslant c_2 R^{1-p} \exp(c_3 R^{a'})$.*

The proof is a direct application of Theorem 2.1. Indeed, for $R > 0$, consider the scalings:

$$I = RI', \qquad \theta = \theta', \qquad G = R^p G', \qquad t = R^{1-p} t'.$$

Then the Hamiltonian $G(\theta, I, t)$, for $(\theta, t, I) \in \mathcal{D}_s \times B_{2R}$, is equivalent to the Hamiltonian $G'(\theta', I', t')$, for $(\theta', t', I') \in \mathcal{D}_s \times B_2$, where $G'(\theta', I', t') = h_p(I') + R^{-p} V(\theta', R^{1-p} t')$. Hence we can apply Theorem 2.1 to the Hamiltonian G', with $\varepsilon = R^{-p}$, $\gamma = (p-1)p^{-1}$, $\rho = 2$ (in Theorem 2.1, constants depend on h_p only through p): there exist positive constants $\varepsilon_0, c_1, c_2, c_3$, that depend on n, p, s, and positive constants a, b that depend only on n, p, such that if $\varepsilon \leqslant \varepsilon_0$, for all solutions $(\theta'(t'), I'(t'))$ of the Hamiltonian system defined by G', if $I'(0) \in B_1$, then $|I'(t') - I'(0)| \leqslant c_1 \varepsilon^b$ for all $|t'| \leqslant c_2 \exp(c_3 \varepsilon^{-a})$. Recalling that $\varepsilon = R^{-p}$, this means that if $R \geqslant R_0 = \varepsilon_0^{-p^{-1}}$, for all $I'(0) \in B_1$ we have $|I'(t') - I'(0)| \leqslant c_1 R^{-b'}$ for all $|t'| \leqslant c_2 \exp(c_3 R^{a'})$ with $b' = pb$ and $a' = pa$. Now scaling back to the original variables, for all $I(0) \in B_R$, we have $|I(t) - I(0)| \leqslant c_1 R^{1-b'}$ for all $|t| \leqslant c_2 R^{1-p} \exp(c_3 R^{a'})$.

Now let us add some comments on the statement of Theorem 3.1. The estimate $|I(t) - I(0)| \leqslant c_1 R^{1-b'}$ is stronger than $|I(t) - I(0)| \leqslant R$ (as in the argument above, the estimate $|I'(t') - I'(0)| \leqslant c_1 \varepsilon^b$ is stronger than $|I'(t') - I'(0)| \leqslant 1$) and hence it is stronger than $I(t) \in B_{2R}$ if $I(0) \in B_R$. Moreover, we have $|t| \leqslant c_2 \exp(c_3' R^{a'}) \leqslant c_2 R^{1-p} \exp(c_3 R^{a'})$ by restricting c_3 to a smaller value c_3' and enlarging R_0 if necessary. So even for the convex case $p = 2$, our statement is more accurate than the statement in [4]. In fact, for $p = 2$, we already explained that we believe that we can choose $a = b = (2n)^{-1}$, in which case the statement of Theorem 3.1 would read $|I(t) - I(0)| \leqslant c_1 R^{1-n^{-1}}$ for all $|t| \leqslant c_2 R^{-1} \exp(c_3 R^{n^{-1}})$, which would be in perfect agreement with the much simpler autonomous case $V(\theta, t) = V(\theta)$ described in [13].

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