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# A note on the r-Whitney numbers of Dowling lattices



## Une note sur le r-nombre de Whitney d'un réseau de Dowling

## Mircea Merca

Department of Mathematics, University of Craiova, 200585 Craiova, Romania

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### ABSTRACT

The complete and elementary symmetric functions are specializations of Schur functions. In this paper, we use this fact to give two identities for the complete and elementary symmetric functions. This result can be used to proving and discovering some algebraic identities involving r-Whitney and other special numbers.

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#### RÉSUMÉ

Les fonctions symétriques complètes et élémentaires sont des spécialisations de fonctions de Schur. Dans cet article, nous utilisons ce fait pour donner deux identités pour les fonctions symétriques complètes et élémentaires. Ce résultat peut être utilisé pour démontrer et découvrir des identités algébriques impliquant les nombres r-Whitney et d'autres nombres spéciaux.

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### 1. Introduction

In 1973, as a generalization of the partition lattice, Dowling [11] introduced a class of geometric lattices based on finite groups, called the Dowling lattice. Given a finite group G of order p > 0, let  $Q_n(G)$  be the Dowling lattice of rank n associated with G. For  $0 \le k \le n$ , the Whitney numbers  $w_p(n,k)$  of the first kind of  $Q_n(G)$  are given by:

$$p^{n}(x)_{n} = \sum_{k=0}^{n} w_{p}(n, k)(px + 1)^{k}$$

and the Whitney numbers  $W_p(n,k)$  of the second kind of  $Q_n(G)$  are given by:

$$(px+1)^n = \sum_{k=0}^n p^k W_p(n,k)(x)_k,$$

where  $(x)_n$  is the falling factorial, i.e.,

$$(x)_n = x(x-1)\cdots(x-n+1),$$

with  $(x)_0 = 1$ . Many properties of Whitney numbers and their combinatorial interpretations can be seen in [4–6,11]. For further information on lattices, see [10,11].

E-mail address: mircea.merca@profinfo.edu.ro.

The r-Stirling numbers of the first kind:

$$\begin{bmatrix} n \\ k \end{bmatrix}$$

count restricted permutations and are defined, for any positive integer r, as the number of permutations of the set  $\{1, 2, ..., n\}$  that have k cycles such that the numbers 1, 2, ..., r are in distinct cycles. The r-Stirling numbers of the second kind:

$$\begin{Bmatrix} n \\ k \end{Bmatrix}$$

are defined as the number of partitions of the set  $\{1, 2, ..., n\}$  into k non-empty disjoint subsets, such that the numbers 1, 2, ..., r are in distinct subsets. Recall that the r-Stirling numbers were introduced into the literature by Broder [7] in 1984 as a generalization of the classical Stirling numbers. Many properties of r-Stirling numbers can be seen in [16,17].

The r-Whitney numbers were introduced in 2010 by Mező [18] as a new class of numbers generalizing the Whitney and r-Stirling numbers. According to [9,18], the nth power of px+r can be expressed in terms of the falling factorial as follows:

$$(px+r)^{n} = \sum_{k=0}^{n} p^{k} W_{p,r}(n,k)(x)_{k},$$

where the coefficients  $W_{p,r}(n,k)$  are called r-Whitney numbers of the second kind. The r-Whitney numbers of the first kind are the coefficients of  $(px+r)^k$  in the reverse relation:

$$p^{n}(x)_{n} = \sum_{k=0}^{n} w_{p,r}(n,k)(px+r)^{k}.$$

We note that the r-Whitney numbers of both kinds may be reduced to the r-Stirling numbers of both kinds by setting p = 1, i.e.,

$$w_{1,r}(n, n-k) = (-1)^k \begin{bmatrix} n+r \\ n+r-k \end{bmatrix}_r$$

and

$$W_{1,r}(n+k,n) = \left\{ \begin{array}{l} n+r+k \\ n+r \end{array} \right\}_r.$$

It is clear that the case r = 1 gives the Whitney numbers of Dowling lattices.

Recently, Cheon [9] proved some algebraic identities for the r-Whitney numbers of both kinds. For example, the r-Whitney numbers of the second kind may be expressed in terms of the Stirling numbers S(n,k) of the second kind:

$$W_{p,r}(n,k) = \sum_{i=k}^{n} {n \choose i} p^{i-k} r^{n-i} S(i,k),$$

a connection between the r-Whitney numbers and r-Stirling numbers of the second kind is given by:

$$W_{p,r}(n,k) = \sum_{i=k}^{n} {n \choose i} p^{i-k} (r-pr)^{n-i} \begin{Bmatrix} i+r \\ k+r \end{Bmatrix}_r$$

and for nonnegative integers  $r \ge s$ , we have the following relation between r- and s-Whitney numbers of the second kind:

$$W_{p,r}(n,k) = \sum_{i=k}^{n} {n \choose i} (r-s)^{n-i} W_{p,s}(i,k).$$

On the other hand, the r-Whitney numbers of the first kind can be expressed in terms of the Stirling numbers s(n,k) of the first kind:

$$w_{p,r}(n,k) = \sum_{i=1}^{n} {i \choose k} p^{n-i} (-r)^{i-k} s(n,i).$$

Motivated by these identities, we give in this paper two relations for the complete and elementary symmetric functions. These are different from those recently presented by Merca [15]. The above identities are very specific cases. This approach can be easily used to prove and discover new algebraic identities involving some special numbers (Jacobi–Stirling numbers, q-Stirling numbers, q-binomial coefficients, r-Whitney numbers or central factorial numbers). The idea is illustrated by several applications on the r-Whitney numbers. A new connection between the r-Whitney numbers and central factorial numbers is presented.

#### 2. Main result

Any positive integer n can be written as a sum of one or more positive integers, i.e.,

$$n = \lambda_1 + \lambda_2 + \dots + \lambda_r. \tag{1}$$

When the order of integers  $\lambda_i$  does not matter, this representation is known as an integer partition [1] and can be rewritten as:

$$n = t_1 + 2t_2 + \cdots + nt_n$$

where each positive integer i appears  $t_i$  times. If the order of integers  $\lambda_i$  is important, then the representation (1) is known as a composition. For

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r$$
,

we have a descending composition. We notice that more often than not there appears the tendency of defining partitions as descending compositions and this is also the convention used in this paper. In order to indicate that:

$$\lambda = [\lambda_1, \lambda_2, \dots, \lambda_r]$$
 or  $\lambda = [1^{t_1} 2^{t_2} \dots n^{t_n}]$ 

is a partition of n, we use the notation  $\lambda \vdash n$ . We denote by  $l(\lambda)$  the number of parts of  $\lambda$ , i.e.,

$$l(\lambda) = r$$
 or  $l(\lambda) = t_1 + t_2 + \cdots + t_n$ .

Let k be a positive integer such that  $k \le n$ . For each partition  $\lambda \vdash k$ , the Schur function  $s_{\lambda}$  in n variables can be defined as the ratio of two  $n \times n$  determinants as follows [14, I.3]:

$$s_{\lambda}(x_1, x_2, \dots, x_n) = \frac{\det(x_i^{\lambda_j + n - j})_{1 \leqslant i, j \leqslant n}}{\det(x_i^{n - j})_{1 \leqslant i, j \leqslant n}},$$

where we consider that  $\lambda_j = 0$  for  $j > l(\lambda)$ . If  $\lambda = [1^k]$ , then  $s_{\lambda}$  is the kth elementary symmetric function  $e_k$ ,

$$e_k(x_1, x_2, \dots, x_n) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} x_{i_1} x_{i_2} \dots x_{i_k}.$$

When  $\lambda = [k]$ ,  $s_{\lambda}$  is the kth complete homogeneous symmetric function  $h_k$ ,

$$h_k(x_1, x_2, \dots, x_n) = \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \dots \leqslant i_k \leqslant n} x_{i_1} x_{i_2} \dots x_{i_k}.$$

In this section, we shall prove:

**Theorem 1.** Let k and n be two positive integers. Then:

$$f_k(t+x_1,t+x_2,\ldots,t+x_n) = \sum_{i=0}^k \binom{n-c_i}{k-i} f_i(x_1,x_2,\ldots,x_n) t^{k-i}$$

and

$$f_k(x_1, x_2, ..., x_n) = \sum_{i=0}^k {n-c_i \choose k-i} f_i(t+x_1, t+x_2, ..., t+x_n)(-t)^{k-i},$$

where  $t, x_1, x_2, \dots, x_n$  are independent variables, f is any of these complete or elementary symmetric functions and:

$$c_i = \begin{cases} i, & \text{if } f_i = e_i, \\ 1 - k, & \text{if } f_i = h_i. \end{cases}$$

**Proof.** To prove the theorem, we use the following Schur function formula [14, p. 47]:

$$s_{\lambda}(x_1+1,\ldots,x_n+1) = \sum_{\mu\subseteq\lambda} d_{\lambda\mu} s_{\mu}(x_1,\ldots,x_n),$$

where:

$$d_{\lambda\mu} = \det\left(\binom{\lambda_i + n - i}{\mu_j + n - j}\right)_{1 \leqslant i, j \leqslant n}.$$

Since  $s_{[1^k]} = e_k$ , this implies:

$$e_k(x_1+1,\ldots,x_n+1) = \sum_{m=0}^k d_{[1^k][1^m]} e_m(x_1,\ldots,x_n).$$
 (2)

Taking into account [8, Theorem 1], it is an easy exercise to show that:

$$d_{[1^k][1^m]} = \binom{n-m}{k-m}.$$

On the other hand, for  $\lambda = [k]$ , we obtain:

$$h_k(x_1+1,\ldots,x_n+1) = \sum_{m=0}^k d_{[k][m]} h_m(x_1,\ldots,x_n),$$
(3)

where:

$$d_{[k][m]} = \binom{n-1+k}{k-m}.$$

Because

$$f_m(tx_1, tx_2, \dots, tx_n) = t^m f_m(x_1, x_2, \dots, x_n),$$

we replace  $x_k$  by  $x_k/t$  in (2) and (3). Thus, the first identity is proved. The second identity follows easily from the first identity.  $\Box$ 

### 3. Applications with r-Whitney numbers

The relationships of the complete and elementary symmetric functions to the r-Whitney numbers:

$$W_{p,r}(n+1, n+1-k) = (-1)^k e_k(r, p+r, 2p+r, ..., np+r)$$

and

$$W_{p,r}(n+k,n) = h_k(r, p+r, 2p+r, ..., np+r)$$

are well known [9].

**Corollary 1.** The r-Whitney numbers may be expressed in terms of the s-Whitney numbers:

1. 
$$w_{p,r}(n,k) = \sum_{i=k}^{n} {i \choose k} w_{p,s}(n,i)(s-r)^{i-k},$$

2. 
$$W_{p,r}(n,k) = \sum_{i=-k}^{n} \binom{n}{i} W_{p,s}(i,k) (r-s)^{n-i}$$
.

**Proof.** By Theorem 1, we obtain:

$$w_{p,r}(n, n-k) = \sum_{i=0}^{k} {n-i \choose k-i} w_{p,s}(n, n-i)(s-r)^{k-i}$$

and

$$W_{p,r}(n+k,n) = \sum_{i=0}^{k} {n+k \choose k-i} W_{p,s}(n+i,n)(r-s)^{k-i}.$$

The proof follows easily.  $\Box$ 

Using this corollary, it is an easy exercise to show that the r-Whitney numbers can be expressed in terms of Whitney numbers and vice versa.

According to [7], the r-Stirling numbers of the first kind are the elementary symmetric functions of the numbers  $r, \ldots, n$ , i.e.,

$$\begin{bmatrix} n+1 \\ n+1-k \end{bmatrix}_r = e_k(r, \dots, n)$$

and the r-Stirling numbers of the second kind are the complete homogeneous symmetric functions of the numbers  $r, \ldots, n$ , i.e.,

$$\begin{Bmatrix} n+k \\ n \end{Bmatrix}_r = h_k(r, \dots, n).$$

**Corollary 2.** The r-Whitney numbers may be expressed in terms of the r-Stirling numbers and vice versa:

1. 
$$w_{p,r}(n,k) = \sum_{i=k}^{n} {i \choose k} {n+r \brack i+r}_r (-p)^{n-i} (pr-r)^{i-k},$$

3. 
$$W_{p,r}(n,k) = \sum_{i=-k}^{n} {n \choose i} {i+r \brace k+r}_r p^{i-k} (r-pr)^{n-i},$$

4. 
$${n+r \brace k+r}_r = \frac{1}{p^{n-k}} \sum_{i=k}^n {n \choose i} W_{p,r}(i,k) (pr-r)^{n-i}$$
.

**Proof.** These identities follow directly from Theorem 1. For the first two identities, we have:

$$\begin{split} w_{p,r}(n,n-k) &= (-1)^k e_k \big(r,p+r,2p+r,\ldots,(n-1)p+r\big) \\ &= (-1)^k \sum_{i=0}^k \binom{n-i}{k-i} e_i \big(pr,pr+p,pr+2p,\ldots,pr+(n-1)p\big) (r-pr)^{k-i} \\ &= (-1)^k \sum_{i=0}^k \binom{n-i}{k-i} e_i (r,r+1,r+2,\ldots,r+n-1) p^i (r-pr)^{k-i} \\ &= \sum_{i=0}^k \binom{n-i}{k-i} e_i (r,r+1,r+2,\ldots,r+n-1) (-p)^i (pr-r)^{k-i} \\ &= \sum_{i=0}^k \binom{n-i}{k-i} e_i (r,r+1,r+2,\ldots,r+n-1) (-p)^i (pr-r)^{k-i} \end{split}$$

and

$$\begin{bmatrix}
n+r \\
n+r-k
\end{bmatrix}_{r} = e_{k}(r, r+1, r+2, ..., r+n-1)$$

$$= \frac{1}{p^{k}} e_{k} (pr, pr+p, pr+2p, ..., pr+p(n-1))$$

$$= \frac{1}{p^{k}} \sum_{i=0}^{k} {n-i \choose k-i} e_{i} (r, p+r, 2p+r, ..., (n-1)p+r) (pr-r)^{k-i}$$

$$= \frac{1}{p^{k}} \sum_{i=0}^{k} {n-i \choose k-i} w_{p,r}(n, n-i) (-1)^{i} (pr-r)^{k-i}.$$

The proof for the last two identities is similar.  $\Box$ 

Recall that the numbers:

$$s(n, n - k) = (-1)^k e_k(1, 2, ..., n - 1)$$

are known as Stirling numbers of the first kind and the numbers:

$$S(n+k,n) = h_k(1, 2, ..., n)$$

are known as Stirling numbers of the second kind. By Theorem 1, we get the following.

**Corollary 3.** The r-Whitney numbers may be expressed in terms of the Stirling numbers and vice versa:

1. 
$$w_{p,r}(n,k) = \sum_{i=1}^{n} {i \choose k} s(n,i) p^{n-i} (-r)^{i-k}$$

2. 
$$s(n,k) = \frac{1}{p^{n-k}} \sum_{i=k}^{n} {i \choose k} w_{p,r}(n,i) r^{i-k},$$

3. 
$$W_{p,r}(n,k) = \sum_{i=k}^{n} \binom{n}{i} S(i,k) p^{i-k} r^{n-i},$$

4. 
$$S(n,k) = \frac{1}{p^{n-k}} \sum_{i=k}^{n} {n \choose i} W_{p,r}(i,k) (-r)^{n-i}$$
.

In Riordan's book [20, pp. 213–217], we see that the central factorial numbers of the first kind are the elementary symmetric functions of the numbers  $\frac{n}{2}-1,\frac{n}{2}-2,\ldots,\frac{n}{2}-(n-1)$ , i.e.,

$$t(n, n-k) = e_k \left(\frac{n}{2} - 1, \frac{n}{2} - 2, \dots, \frac{n}{2} - (n-1)\right)$$

and the central factorial numbers of the second kind are the complete symmetric functions of the numbers  $\frac{n}{2} - 0$ ,  $\frac{n}{2} - 1$ , ...,  $\frac{n}{2} - n$ , i.e.,

$$T(n+k,n) = h_k \left(\frac{n}{2} - 0, \frac{n}{2} - 1, \dots, \frac{n}{2} - n\right).$$

**Corollary 4.** The r-Whitney numbers may be expressed in terms of the central factorial numbers and vice versa:

1. 
$$w_{p,r}(n,k) = \sum_{i=1}^{n} {i \choose k} t(n+1,i+1) p^{n-i} \left(-r - \frac{(n-1)p}{2}\right)^{i-k}$$
,

2. 
$$t(n+1, k+1) = \frac{1}{p^{n-k}} \sum_{i=k}^{n} {i \choose k} w_{p,r}(n, i) \left(r + \frac{(n-1)p}{2}\right)^{i-k}$$
,

3. 
$$W_{p,r}(n,k) = \sum_{i=k}^{n} {n \choose i} T(i,k) (-p)^{i-k} \left(r + \frac{kp}{2}\right)^{n-i},$$

4. 
$$T(n,k) = \frac{1}{p^{n-k}} \sum_{i=k}^{n} {n \choose i} W_{p,r}(i,k) (-1)^{i-k} \left(r + \frac{kp}{2}\right)^{n-i}$$
.

Proof. We have

$$\begin{split} w_{p,r}(n-1,n-1-k) &= (-1)^k e_k \big(r,p+r,2p+r,\dots,(n-2)p+r\big) \\ &= (-1)^k \sum_{i=0}^k \binom{n-1-i}{k-i} e_i \bigg(p-\frac{pn}{2},2p-\frac{pn}{2},\dots,(n-1)p-\frac{pn}{2}\bigg) \bigg(r-p+\frac{pn}{2}\bigg)^{k-i} \\ &= \sum_{i=0}^k \binom{n-1-i}{k-i} e_i \bigg(\frac{n}{2}-1,\frac{n}{2}-2,\dots,\frac{n}{2}-(n-1)\bigg) p^i \bigg(-r-\frac{(n-2)p}{2}\bigg)^{k-i} \\ &= \sum_{i=0}^k \binom{n-1-i}{k-i} t(n,n-i) p^i \bigg(-r-\frac{(n-2)p}{2}\bigg)^{k-i} \end{split}$$

and

$$t(n, n - k) = e_k \left( \frac{n}{2} - 1, \frac{n}{2} - 2, \dots, \frac{n}{2} - (n - 1) \right)$$
$$= \frac{1}{p^k} e_k \left( \frac{pn}{2} - p, \frac{pn}{2} - 2p, \dots, \frac{pn}{2} - (n - 1)p \right)$$

$$\begin{split} &= \frac{1}{p^k} \sum_{i=0}^k \binom{n-1-i}{k-i} e_i \Big( -r, -p-r, -2p-r, \dots, -(n-2)p-r \Big) \left( r + \frac{(n-2)p}{2} \right)^{k-i} \\ &= \frac{1}{p^k} \sum_{i=0}^k \binom{n-1-i}{k-i} w_{p,r} (n-1, n-1-i) \left( r + \frac{(n-2)p}{2} \right)^{k-i}. \end{split}$$

We deduce that

$$w_{p,r}(n, n-k) = \sum_{i=0}^{k} \binom{n-i}{k-i} t(n+1, n+1-i) p^{i} \left(-r - \frac{(n-1)p}{2}\right)^{k-i}$$

and

$$t(n+1, n+1-k) = \frac{1}{p^k} \sum_{i=0}^k \binom{n-i}{k-i} w_{p,r}(n, n-i) \left(r + \frac{(n-1)p}{2}\right)^{k-i}.$$

Thus, the first two identities follow immediately. The proof for the last two identities is similar.  $\Box$ 

The identities presented in these corollaries are just some of the consequences of Theorem 1. Connections between r-Stirling numbers and Whitney numbers, Whitney numbers and Stirling numbers, or r-Stirling numbers and Stirling numbers can be immediately derived from our theorem. According to [2,3,12,13,19], the similar identities involving Legendre–Stirling numbers, Jacobi–Stirling numbers and central factorial numbers with odd indices can be obtained as well.

Recently, Merca [15] published new identities involving r-Whitney numbers. These identities are different from those presented in this paper.

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