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Functional analysis

Sums of Murray–von Neumann equivalent operators



Sommets d'opérateurs Murray–von Neumann équivalents

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ABSTRACT

Let A, B be two Hilbert space positive operators such that $1 \geq \|B\| \neq 0$ and the positive part of $A - I$ satisfies $\text{Tr}(A - I)_+ = \infty$. Then $A = \sum_{n=1}^{\infty} B_n$, where $B_n \sim B$ for all n . ($X \sim Y$ means $X = TT^*$ and $Y = T^*T$.) This extends a 2009 result of Kaftal, Ng, and Zhang for sums of projections.

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R É S U M É

Si A est un opérateur positif tel que la partie positive de $A - I$ vérifie $\text{Tr}(A - I)_+ = \infty$, alors A est une somme de projections de rangs infinis. Ce résultat, obtenu en 2009 par Kaftal, Ng et Zhang, est étendu dans cette note aux sommes d'opérateurs Murray–von Neumann équivalents à une contraction positive arbitraire.

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1. Infinite sums of projections and contractions

By operator, we mean a bounded linear operator on an infinite dimensional, separable (real or complex) Hilbert space \mathcal{H} . A positive operator is a self-adjoint operator whose spectrum lies on the non-negative half-line. The simplest positive operators are projections. If a projection P has a range of infinite dimension, we say that P is infinite. Two projections P, Q are equivalent, $P \sim Q$, if their ranges have the same dimension, thus if $P = TT^*$ and $Q = T^*T$ for some T (here, a partial isometry). This classical relation (Murray–von Neumann) makes sense for positive operators: $X \sim Y$ merely means that, restricting X and Y to their supports, these two restrictions are unitarily equivalent. A self-adjoint operator X can be decomposed into positive and negative parts, $X = X_+ - X_-$. Then $\text{Tr} X_+, \text{Tr} X_- \in [0, \infty]$. In [5] Kaftal, Ng, and Zhang showed that a positive operator is a strong sum of projections under a rather optimal assumption, improving an earlier result of Dykema et al. [4]. Their result entails the following theorem. I denotes the identity on \mathcal{H} or on any subspace and \mathbb{N}^* is the set of positive integers.

Theorem 1.1. *Let A be a positive operator such that $\text{Tr}(A - I)_+ = \infty$. Then, there exists a family of infinite projections Q_j , $j \in \mathbb{N}^*$, such that:*

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$$A = \sum_{j=1}^{\infty} Q_j.$$

The sum is a strong limit, norm convergence cannot hold. Since infinite projections are equivalent, it seems natural to ask whether [Theorem 1.1](#) is a special case of a more general result for sums of equivalent copies of any given positive contraction. This generalization does hold.

Theorem 1.2. *Let A be a positive operator such that $\text{Tr}(A - I)_+ = \infty$ and let $B \neq 0$ be a positive contraction. Then, there exists a decomposition:*

$$A = \sum_{j=1}^{\infty} B_j$$

where $B_j \sim B$ for all j .

It would be interesting to obtain a similar result involving the usual unitary equivalence relation, B is unitarily equivalent to each B_j . Extra-assumptions on A and B would then be necessary; reasonable candidates are: (i) A is nonsingular and (ii) the essential spectrum of B contains 0.

Kaftal, Ng, and Zhang established [Theorem 1.1](#) in the more general setting of factors (see also [\[6\]](#) for results in the setting of multiplier algebras). [Theorem 1.2](#) probably holds for factors too; however, this is beyond the scope of our methods. Indeed, our approach makes use of the following key block-matrix decomposition from [\[2, Lemma 3.4\]](#); this decomposition is not relevant in the setting of a general factor if the blocks do not belong to the factor too. Let \mathbb{M}_n stand for the space of n -by- n matrices and let \mathbb{M}_n^+ be its positive semi-definite part. For every matrix in \mathbb{M}_{n+m}^+ written in blocks, we have a decomposition:

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = U \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} V^* \quad (1.1)$$

for some unitaries $U, V \in \mathbb{M}_{n+m}$. This is a simple but powerful fact, see [\[2,3\]](#).

2. Main steps of the proof of [Theorem 1.2](#)

This note is essentially self-contained, the next series of lemma easily follows from the well-known facts. Details, including a new proof of [Theorem 1.1](#), and further consequences of decomposition [\(1.1\)](#) will be given in a forthcoming work. By operator, we always mean an operator on \mathcal{H} unless otherwise specified. We will use block-matrix representations with respect to infinite direct sums. We say that an infinite orthogonal direct sum:

$$\mathcal{H} = \bigoplus_{k=1}^{\infty} \mathcal{H}_k$$

is a *standard decomposition* if $\dim \mathcal{H}_k = \infty$ for all $k \in \mathbb{N}^*$. An operator A then admits an operator matrix representation $A = (A_{i,j})$ with respect to this standard decomposition, where $A_{i,j} : \mathcal{H}_j \rightarrow \mathcal{H}_i$ is defined in an obvious way, $i, j \in \mathbb{N}^*$. The first lemma adapts decomposition [\(1.1\)](#) to operators on the Hilbert space \mathcal{H} .

Lemma 2.1. *Let A be a positive operator with a matrix representation $A = (A_{i,j})$ for some standard decomposition $\mathcal{H} = \bigoplus_{k=1}^{\infty} \mathcal{H}_k$ and all diagonal entries $A_{k,k}$ nonsingular. Then:*

$$A = \sum_{k=1}^{\infty} V_k A_{k,k} V_k^*$$

for some isometries $V_k : \mathcal{H}_k \rightarrow \mathcal{H}$, $k \in \mathbb{N}^*$.

The second idea of the proof consists in combining the above lemma with the pinching theorem from [\[1\]](#). We denote by $W_e(A)$ the essential numerical range of an operator A and refer to [\[1\]](#) for various equivalent definitions and basic properties. The pinching theorem holds for complex and real Hilbert spaces. It may be stated as the next lemma. \mathcal{D} stands for the closed unit disc of the complex plane. Given an operator X on \mathcal{H} and an operator Y on a subspace $\mathcal{S} \subset \mathcal{H}$, the notation $Y \simeq X$ means that $Y = U^* X U$ for an onto isometry $U : \mathcal{S} \rightarrow \mathcal{H}$.

Lemma 2.2. *Let A be an operator such that $W_e(A) \supset \mathcal{D}$ and let $\{X_i\}_{i=1}^{\infty}$ be a sequence of operators such that $\sup_i \|X_i\| < 1$. Then, there exists a standard decomposition of \mathcal{H} for which $A = (A_{i,j})$ with diagonal blocks $A_{i,i} \simeq X_i$ for all $i \in \mathbb{N}^*$.*

The next lemma is an easy consequence for positive operators.

Lemma 2.3. *Let A be a positive operator such that $W_e(A) \supset (0, a)$ for some $a > 1$, and let $\{X_i\}_{i=1}^\infty$ be a sequence of positive operators such that $\beta \leq X_i \leq 1$ for some $\beta > 0$ and all $i \in \mathbb{N}^*$. Then, there exists a standard decomposition of \mathcal{H} for which $A = (A_{i,j})$ with diagonal blocks $A_{i,i} \simeq X_i$ for all $i \in \mathbb{N}^*$.*

If A is a positive operator whose essential norm satisfies $\|A\|_e > 1$, then we may decompose A in a sum of two positive operators with essential numerical ranges containing an interval $(0, a)$ for some $a > 1$. This simple observation combined with Lemma 2.1 and Lemma 2.3 yields our last lemma. It is convenient to introduce a notation. The support of a positive operator X is $\text{supp } X = \mathcal{H} \ominus \ker X$ and, given $\beta > 0$, we write $X \geq^s \beta$ if $X|_{\text{supp } X} \geq \beta$, i.e., if $X \geq \beta R_X$ where R_X is the range projection.

Lemma 2.4. *Let A be a positive operator such that $\|A\|_e > 1$ and let $\{B_j\}_{j=1}^\infty$ be a family of positive contractions and $\beta > 0$ such that $B_j \geq^s \beta$ for all j . Then, there exists a decomposition:*

$$A = \sum_{j=1}^\infty C_j$$

where $C_j \sim B_j$ for all j .

In case of all the B_j have infinite ranks, Lemma 2.4 is a straightforward consequence of Lemma 2.1 and Lemma 2.3. The general case follows by considering direct sums among the B_j in order to have infinite rank operators.

Theorem 1.2 follows from its special case Theorem 1.1 combined with Lemma 2.4. In fact, Lemma 2.4 allows one to cover the case of contractions B such that $\|Bh\| < \|h\|$ for all nonzero h , and the use of Theorem 1.1 is required for the case of B with 1 as eigenvalue. We turn to the proof of Theorem 1.2.

Proof. (a) Thanks to the spectral resolution of B , we have a direct sum decomposition, for some $m \in \mathbb{N}^* \cup \infty$,

$$\mathcal{H} = \mathcal{H}_0 \oplus \left(\bigoplus_{k=1}^m \mathcal{H}_k \right)$$

where $\dim \mathcal{H}_k = \infty$ for all $k \geq 1$ and either $\dim \mathcal{H}_0 = 0$, that is \mathcal{H}_0 is vacuous, or $1 \leq \dim \mathcal{H}_0 \leq \infty$, and such that, with respect to this decomposition,

$$B = B_0 \oplus \left(\bigoplus_{k=1}^m B_k \right)$$

with $B_0 = I$ on \mathcal{H}_0 (B_0 is vacuous if so is \mathcal{H}_0), and $B_k : \mathcal{H}_k \rightarrow \mathcal{H}_k$ satisfies $1 > \|B_k\|$ and $B_k \geq^s \beta_k$ for some $\beta_k > 0$, $1 \leq k \leq m$.

(b) We also have a direct sum decomposition, with the same $m \in \mathbb{N}^* \cup \infty$ as in (a),

$$\mathcal{H} = \mathcal{H}'_0 \oplus \left(\bigoplus_{k=1}^m \mathcal{H}'_k \right)$$

where $\dim \mathcal{H}'_k = \infty$ for integers k , $1 \leq k \leq m$, and either \mathcal{H}'_0 is vacuous if so is \mathcal{H}_0 or $\dim \mathcal{H}'_0 = \infty$, and such that, with respect to this decomposition,

$$A = A'_0 \oplus \left(\bigoplus_{k=1}^m A'_k \right)$$

such that $\text{Tr}(A'_k - I)_+ = \infty$ for all integers k , $1 \leq k \leq m$, and also $\text{Tr}(A'_0 - I)_+ = \infty$ if \mathcal{H}'_0 is not vacuous.

(c) If $\mathcal{H}_0, \mathcal{H}'_0$ are not vacuous, Theorem 1.1 yields a family of projections $\{P_j\}_{j=1}^\infty$ of rank $\dim \mathcal{H}_0$ such that $A'_0 = \sum_{j=1}^\infty P_j$, and hence:

$$A'_0 = \sum_{j=1}^\infty V_j^0 B_0 V_j^{0*} \tag{2.1}$$

for some isometries $V_j^0 : \mathcal{H}_0 \rightarrow \mathcal{H}'_0$, $j \in \mathbb{N}^*$.

(d) Let $k \geq 1$. By Lemma 2.4 applied to A'_k and B_k , there exists a family of isometries $V_j^k : \mathcal{H}_k \rightarrow \mathcal{H}'_k$, $1 \leq j < \infty$ such that:

$$A'_k = \sum_{j=1}^{\infty} V_j^k B_k V_j^{k*}. \quad (2.2)$$

(e) Putting together (2.1) (if not vacuous) and each Eq. (2.2), $k \geq 1$, and defining some isometries $V_j : \mathcal{H} \rightarrow \mathcal{H}$, for $j \in \mathbb{N}^*$, by $V_j|_{\mathcal{H}_k} = V_j^k$ one gets:

$$A = \sum_{j=1}^{\infty} V_j B V_j^*$$

which establishes the theorem. \square

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