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Spectral stability results for higher-order operators under perturbations of the domain





Stabilité spectrale des opérateurs d'ordre supérieur pour des perturbations du domaine

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ABSTRACT

We analyze the spectral behavior of higher-order elliptic operators when the domain is perturbed. We provide general spectral stability results for Dirichlet and Neumann boundary conditions. Moreover, we study the bi-harmonic operator with the so-called intermediate boundary conditions. We give special attention to this last case and analyze its behavior when the boundary of the domain has some oscillatory behavior. We will show that there is a critical oscillatory behavior and that the limit problem depends on whether we are above, below or just sitting on this critical value.

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RÉSUMÉ

Nous analysons le comportement spectral des opérateurs elliptiques d'ordre supérieur lorsque le domaine est perturbé. Nous fournissons des résultats généraux de stabilité spectrale, pour les conditions de Dirichlet et de Neumann. Par ailleurs, nous étudions l'opérateur bi-harmonique avec les conditions aux limites dites intermédiaires. Nous accordons une attention particulière à ce dernier cas et analysons son comportement lorsque la frontière du domaine a un comportement oscillatoire. Nous allons montrer qu'il existe un comportement oscillatoire critique et que le problème à la limite dépend de ce que nous sommes au-dessus, en dessous ou précisement sur cette valeur critique.

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Dans cette note, nous considérons des opérateurs elliptiques autoadjoints d'ordre supérieur de type général (6), avec $m \ge 2$, et les coefficients $A_{\alpha\beta}$ satisfaisant la condition de coercitivité (2). Nous étudions leur comportement spectral lorsque le domaine subit une perturbation. Pour cela, nous considérons la formulation faible de ces problèmes sur la base de la forme bilinéaire (1) avec des conditions de Dirichlet ou de Neumann, ou encore avec les conditions dites intermédiaires. Cela revient à envisager la forme bilinéaire (1) définie, respectivement, sur l'espace $V(\Omega) = W_0^{m,2}(\Omega)$, $V(\Omega) = W^{m,2}(\Omega)$ ou $V(\Omega) = W^{m,2}(\Omega) \cap W_0^{k,2}(\Omega)$ pour un nombre entier k, avec $1 \le k \le m - 1$.

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Nous allons analyser les propriétés discrètes de convergence de ces opérateurs, telles que décrites dans la proposition 2.2. (Voir [16–18,11]; voir aussi [3] pour une application de ces techniques à des problèmes de perturbation de domaine.)

Nous obtenons la condition (C), voir définition 2.3, qui garantit la convergence des résolvantes dans le sens de la définition 2.1 dans une situation assez générale, ce qui implique le convergence spectrale des opérateurs. Cette méthode est appliquée pour obtenir une condition assez simple qui garantit la stabilité spectrale sous les conditions de Neumann, dans l'esprit des résultats obtenus dans [1,2].

Pour le cas des conditions aux limites intermédiaires, nous limitons notre analyse à l'opérateur bi-harmonique, c'est-àdire que nous considérons la forme bilinéaire (8) dans l'espace $V(\Omega) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Nous considérons également la perturbation qui consiste à déformer le bord du domaine Ω de façon oscillatoire. Nous voyons que la «force» des oscillations permettra de déterminer les conditions aux limites à la limite. Ainsi, pour les oscillations lentes, on obtient le résultat que le problème à la limite est l'opérateur biharmonique, avec conditions aux limites également de type intermédiaire. Pour des oscillations très fortes, nous trouvons que le problème à la limite est de nouveau l'opérateur biharmonique, mais avec des conditions aux limites de Dirichlet. En outre, il existe un comportement oscillatoire critique pour lequel le problème à la limite est différent et contient un terme supplémentaire dans la condition aux limites, qui est lié à la géométrie des oscillations. Le problème à la limite dans ce cas est obtenu par la méthode dite d'éclatement périodique de l'opérateur, très liée à l'analyse de [12].

1. Introduction and notation

For all open sets Ω in \mathbb{R}^N , $N \in \mathbb{N}$, $N \ge 2$, we consider the following bilinear form in Ω :

$$Q_{\Omega}(u,v) = \sum_{|\alpha| = |\beta| = m} \int_{\Omega} A_{\alpha\beta} D^{\alpha} u D^{\beta} v \, \mathrm{d}x + \int_{\Omega} u v \, \mathrm{d}x, \quad u, v \in W^{m,2}(\Omega),$$
(1)

where $\alpha = (\alpha_1, \dots, \alpha_N)$, $\beta = (\beta_1, \dots, \beta_N)$ are multi indices, $|\cdot|$ is the trace of the multi index and the functions $A_{\alpha\beta}$ are bounded measurable real-valued functions defined on the entire \mathbb{R}^N satisfying $A_{\alpha\beta} = A_{\beta\alpha}$ and the coercivity condition:

$$\sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x)\xi_{\alpha}\xi_{\beta} \ge \theta \sum_{|\alpha|=m} |\xi_{\alpha}|^2,$$
(2)

for all $x \in \mathbb{R}^N$, $\xi = (\xi_\alpha)_{|\alpha|=m}$ with $\xi_\alpha \in \mathbb{R}$. As usual, we denote by $W^{m,2}(\Omega)$ the Sobolev space of real-valued functions in $L^{2}(\Omega)$, which have distributional derivatives of order *m* in $L^{2}(\Omega)$, endowed with the norm:

$$\|u\|_{W^{m,2}(\Omega)}^{2} = \|u\|_{L^{2}(\Omega)}^{2} + \sum_{|\alpha|=m} \|D^{\alpha}u\|_{L^{2}(\Omega)}^{2}.$$
(3)

We denote by $W_0^{m,2}(\Omega)$ the closure in $W^{m,2}(\Omega)$ of the space of the C^{∞} -functions with compact support in Ω . We also denote by $Q_{\Omega}(u) = Q_{\Omega}(u, u)$, the quadratic form associated with the bilinear form Q_{Ω} and observe that by the bound-edness of the coefficients $A_{\alpha\beta}$ and condition (2), the bilinear form Q_{Ω} is in fact a scalar product in $W^{m,2}(\Omega)$ and the

corresponding norm $Q_{\Omega}^{1/2}(\cdot)$ is equivalent to the Sobolev norm (3). Let $V(\Omega)$ be a closed linear subspace of $W^{m,2}(\Omega)$ satisfying $W_0^{m,2}(\Omega) \subset V(\Omega)$. We recall that since $V(\Omega)$, endowed with the norm $Q_{\Omega}^{1/2}(\cdot)$, is complete, there exists a uniquely determined non-negative self-adjoint operator $H_{V(\Omega)}$ such that $\text{Dom } H_{V(\Omega)}^{1/2} = V(\Omega)$ and $Q_{\Omega}(u, v) = \langle H_{V(\Omega)}^{1/2} u, H_{V(\Omega)}^{1/2} u \rangle_{L^{2}(\Omega)}$ for all $u, v \in V(\Omega)$. In particular, a function u belongs to the domain of $H_{V(\Omega)}$ if and only if $u \in V(\Omega)$ and there exists $f \in L^{2}(\Omega)$ such that:

$$Q_{\Omega}(u,v) = \langle f, v \rangle_{L^{2}(\Omega)}, \quad \forall v \in V(\Omega).$$
(4)

Clearly, $H_{V(\Omega)}u = f$. Eq. (4) is the weak formulation of the problem:

$$\begin{cases} Lu = f, & \text{in } \Omega, \\ BC_{V(\Omega)}u = 0, & \text{on } \partial\Omega, \end{cases}$$
(5)

where *L* is the differential operator:

$$Lu = (-1)^m \sum_{|\alpha| = |\beta| = m} D^{\alpha} \left(A_{\alpha\beta}(x) D^{\beta} u \right) + u, \tag{6}$$

and $BC_{V(\Omega)}u = 0$ are homogeneous boundary conditions that depend on the choice of the space $V(\Omega)$.

We recall that if the embedding $V(\Omega) \subset L^2(\Omega)$ is compact, then the operator $H_{V(\Omega)}$ has compact resolvent. In this case, the spectrum of $H_{V(\Omega)}$ is discrete and consists of a sequence of eigenvalues $\lambda_n[V(\Omega)]$ of finite multiplicity, which can be represented by means of the Min-Max Principle:

$$\lambda_n \left[V(\Omega) \right] = \inf_{\substack{E \subset V(\Omega) \\ \dim E = n}} \sup_{\substack{u \in E \\ u \neq 0}} \frac{Q_\Omega(u)}{\|u\|_{L^2(\Omega)}^2}.$$
(7)

Correspondingly, there exists an orthonormal basis in $L^2(\Omega)$ of eigenfunctions $\varphi_n[V(\Omega)]$ associated with the eigenvalues $\lambda_n[V(\Omega)].$

In this paper, we provide a condition that allows us to describe the behavior of the solutions of problem (5), as well as of $\lambda_n[V(\Omega)]$ and $\varphi_n[V(\Omega)]$, under perturbations of Ω . Our condition is general enough to allow the study of non-uniform classes of domain perturbations, in which case the parameters describing the boundaries of the domains are not required to satisfy uniform bounds. This is done by using the notion of \mathcal{E} -compact convergence, see [17,18,11]. See also [3] for an application of these techniques to domain perturbation problems. Our condition provides a unified approach to domain perturbation problems embracing the cases of Dirichlet, Neumann and intermediate homogeneous boundary conditions. In the Dirichlet case, our condition in fact reduces to the celebrated notion of Mosco convergence, see, e.g., [6]. In the case of Neumann boundary conditions, it is a natural extension to higher-order operators of the condition provided in [2] for the Neumann Laplacian. Finally, it also allows the study of intermediate boundary conditions in domains subject to perturbations of oscillatory type.

We refer to [5] and references therein for a pioneer discussion on the stability properties under the three types of boundary conditions, including an analysis of the so-called Babuška-Sapondzhyan paradox. We also refer to [14] for a further discussion on the paradox and [15] for a general reference in this type of problems. We mention that sharp stability estimates for the eigenvalues of higher-order operators subject to Dirichlet and Neumann boundary conditions have been recently proved in [10] where uniform classes of domain perturbations have been considered (see also [9] for related results); moreover, in [7,8], further restrictions on the classes of open sets allow obtaining also analyticity results.

2. A general stability theorem

Let Ω and $V(\Omega)$ be fixed as in the previous section. For all sufficiently small $\epsilon > 0$, we consider perturbations Ω_{ϵ} , $V(\Omega_{\epsilon})$ of Ω , $V(\Omega)$ respectively, where Ω_{ϵ} are open sets and $V(\Omega_{\epsilon})$ are the corresponding spaces of functions defined on Ω_{ϵ} . We assume that the coefficients $A_{\alpha\beta}$ are functions defined in the whole of \mathbb{R}^N not depending on ϵ , and that the operators $H(V(\Omega))$ and $H(V(\Omega_{\epsilon}))$ have compact resolvents.

We denote by \mathcal{E} the extension-by-zero operator, which means that given a real-valued function u defined on some set in \mathbb{R}^N , $\mathcal{E}u$ is the function extended by zero outside the given set. Clearly, for each $\epsilon > 0$, \mathcal{E} can be thought as an operator acting from $L^2(\Omega)$ to $L^2(\Omega_{\epsilon})$, consisting in extending the function by zero to all of \mathbb{R}^N and then restricting it to Ω_{ϵ} . As a matter of fact, this operator \mathcal{E} will be the key to compare functions and operators defined on Ω and Ω_{ϵ} . The following concepts and definitions go back to the works of F. Stummel, see [16] and G. Vainikko, see [17,18] among others. We also refer to [11,3].

Definition 2.1. i) We say that $v_{\epsilon} \in L^2(\Omega_{\epsilon})$ \mathcal{E} -converges to $v \in L^2(\Omega)$ if $||v_{\epsilon} - \mathcal{E}v||_{L^2(\Omega_{\epsilon})} \to 0$ as $\epsilon \to 0$. We write this as $v_{\epsilon} \xrightarrow{\mathcal{E}} v.$

ii) The family of bounded linear operators $B_{\epsilon} \in \mathcal{L}(L^2(\Omega_{\epsilon}))$ \mathcal{EE} -converges to $B \in \mathcal{L}(L^2(\Omega))$ if $B_{\epsilon}v_{\epsilon} \xrightarrow{\mathcal{E}} Bv$ whenever $v_{\epsilon} \xrightarrow{\mathcal{E}} v$. We write this as $B_{\epsilon} \stackrel{\mathcal{E}\mathcal{E}}{\to} B$.

iii) The family of bounded linear and compact operators $B_{\epsilon} \in \mathcal{L}(L^2(\Omega_{\epsilon}))$ \mathcal{E} -compact converges to $B \in \mathcal{L}(L^2(\Omega))$ if $B_{\epsilon} \xrightarrow{\mathcal{E}\mathcal{E}} B$ and for any family of functions $v_{\epsilon} \in L^2(\Omega_{\epsilon})$ with $\|v_{\epsilon}\|_{L^2(\Omega_{\epsilon})} \leq 1$ there exists a subsequence, denoted by v_{ϵ} again, and a function $w \in L^2(\Omega)$ such that $B_{\epsilon} v_{\epsilon} \xrightarrow{\mathcal{E}} w$. We write $B_{\epsilon} \xrightarrow{\mathcal{C}} B$.

The following result relates the \mathcal{E} -compact convergence of a family of operators to their spectral convergence. By this, we mean the convergence of eigenvalues and the associated spectral projections, see [2, Section 2.1]

Proposition 2.2. Assume the operator \mathcal{E} satisfies the condition $\|\mathcal{E}u\|_{L^2(\Omega_{\epsilon})} \to \|u\|_{L^2(\Omega)}$ for each $u \in L^2(\Omega)$. If $B_{\epsilon} \in \mathcal{L}(L^2(\Omega_{\epsilon}))$ are compact and $B_{\epsilon} \xrightarrow{C} B$ then we have the spectral convergence of B_{ϵ} to B.

The following condition on Ω_{ϵ} and $V(\Omega_{\epsilon})$ guarantees that $H(V(\Omega_{\epsilon}))^{-1}$ is \mathcal{E} -compact convergent to $H(V(\Omega))^{-1}$ in the sense of Definition 2.1.

Definition 2.3. (Condition C) Given the family of open sets $\{\Omega_{\epsilon}\}_{0 < \epsilon \leq \epsilon_0}$, and Ω in \mathbb{R}^N and corresponding elliptic operators $H(V(\Omega_{\epsilon})), H(V(\Omega))$ defined on $\Omega_{\epsilon}, \Omega$ respectively, we say that condition (C) is satisfied if for each $\epsilon > 0$ there exists an open set $K_{\epsilon} \subset \Omega \cap \Omega_{\epsilon}$ such that $|\Omega \setminus K_{\epsilon}| \to 0$ as $\epsilon \to 0$ and such that the following conditions are satisfied: (C1) if $v_{\epsilon} \in V(\Omega_{\epsilon})$ and $\sup_{\epsilon>0} \|v_{\epsilon}\|_{W^{m,2}(\Omega_{\epsilon})} < \infty$ then $\lim_{\epsilon\to 0} \|v_{\epsilon}\|_{L^{2}(\Omega_{\epsilon}\setminus K_{\epsilon})} = 0$; (C2) for each $\epsilon > 0$, there exists an operator T_{ϵ} from $V(\Omega)$ to $V(\Omega_{\epsilon})$ such that:

(i)
$$\lim_{\epsilon \to 0} \|T_{\epsilon}\varphi - \varphi\|_{W^{m,2}(K_{\epsilon})} = 0, \quad (ii) \lim_{\epsilon \to 0} \|T_{\epsilon}\varphi\|_{W^{m,2}(\Omega_{\epsilon} \setminus K_{\epsilon})} = 0, \quad (iii) \sup_{\epsilon > 0} \|T_{\epsilon}\varphi\|_{L^{2}(\Omega_{\epsilon})} < \infty;$$

(C3) for each $\epsilon > 0$, there exists an operator E_{ϵ} from $V(\Omega_{\epsilon})$ to $W^{m,2}(\Omega)$ such that:

- (i) if $v_{\epsilon} \in V(\Omega_{\epsilon})$ is such that $\sup_{\epsilon>0} \|v_{\epsilon}\|_{W^{m,2}(\Omega_{\epsilon})} < \infty$ then $\lim_{\epsilon\to 0} \|E_{\epsilon}v_{\epsilon} v_{\epsilon}\|_{W^{m,2}(K_{\epsilon})} = 0$,
- (ii) $\sup_{\epsilon>0} \sup_{\substack{v \in V(\Omega_{\epsilon}) \\ v \neq 0}} \frac{\|E_{\epsilon}v\|_{W^{m,2}(\Omega)}}{\|v\|_{W^{m,2}(\Omega_{\epsilon})}} < \infty,$
- (iii) if $v_{\epsilon} \in V(\Omega_{\epsilon})$ satisfies $\sup_{\epsilon>0} \|v_{\epsilon}\|_{W^{m,2}(\Omega_{\epsilon})} < \infty$ and $E_{\epsilon}v_{\epsilon} \to v$ in $L^{2}(\Omega)$, then $v \in V(\Omega)$.

Remark 1. i) Consider the simpler but very important case where $\Omega \subset \Omega_{\epsilon}$ for all $\epsilon > 0$ and set $K_{\epsilon} = \Omega$ for all $\epsilon > 0$ (in particular $|\Omega \setminus K_{\epsilon}| = 0$). Assume that Ω is sufficiently regular to guarantee the existence of a bounded extension operator from $W^{m,2}(\Omega)$ to $W^{m,2}(\mathbb{R}^N)$. Then the construction of the operator T_{ϵ} from condition (C2) is a key point. For instance, in case of Neumann boundary conditions, the extension operator may serve as operator T_{ϵ} . In case of Dirichlet boundary conditions, the extension by 0 will do it. The case of intermediate boundary conditions is very delicate, as we will see in Section 4. As far as the operator E_{ϵ} is concerned, one may try to use the restriction operator.

ii) Note that condition (C1) can be interpreted as follows: the strip $\Omega_{\epsilon} \setminus K_{\epsilon}$ has to vanish in such a way to prevent 'energy concentration' outside Ω . On the other hand, condition (C3) can be interpreted as follows: the deformation Ω_{ϵ} has to be regular enough to preserve boundary conditions.

We now prove the following general statement.

Theorem 2.4. If condition (C) is satisfied then $H_{\Omega_{\epsilon}}^{-1} \xrightarrow{\mathcal{C}} H_{\Omega}^{-1}$.

Remark 2. If we consider Dirichlet boundary conditions, that is we choose the space $V(\Omega_{\epsilon}) = W_0^{m,2}(\Omega_{\epsilon})$, there is a condition to guarantee the spectral convergence, which is the Mosco convergence of the spaces $W_0^{m,2}(\Omega_{\epsilon})$ to $W_0^{m,2}(\Omega)$, see e.g. [6, Chap. 4]. It is not difficult to see that Mosco convergence implies condition (C).

3. Neumann boundary conditions

We now consider Neumann boundary conditions, which means that we choose $V(\Omega_{\epsilon}) = W^{m,2}(\Omega_{\epsilon})$. We recall that if Ω is bounded and has a continuous boundary (that is, the domain is locally the subgraph of a continuous function), then the Sobolev space $W^{m,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$. Thus, as it is explained in Section 2, the operator $H_{W^{m,2}(\Omega)}$ is well-defined and has compact resolvent.

The following theorem is in fact a generalization to higher-order operators of the results from [1] and [2, Prop. 2.3].

Theorem 1. Let Ω be a bounded open set in \mathbb{R}^N of class $C^{0,1}$ and Ω_{ϵ} , with $\epsilon > 0$, be bounded open sets in \mathbb{R}^N of class C^0 . Assume there exists a sequence $\rho_{\epsilon} > 0$ such that $\rho_{\epsilon} \xrightarrow{\epsilon \to 0} 0$ with the property that if $K_{\epsilon} = \{x \in \Omega : d(x, \partial \Omega) > \rho_{\epsilon}\}$ then $K_{\epsilon} \subset \Omega_{\epsilon}$. Assume also one of the following two equivalent conditions:

i) if $v_{\epsilon} \in W^{m,2}(\Omega_{\epsilon})$ and $\sup_{\epsilon>0} \|v_{\epsilon}\|_{W^{m,2}(\Omega_{\epsilon})} < \infty$ then $\lim_{\epsilon \to 0} \|v_{\epsilon}\|_{L^{2}(\Omega_{\epsilon} \setminus K_{\epsilon})} = 0$; ii) $\lim_{\epsilon \to 0} \tau_{\epsilon} = \infty$, where:

$$\tau_{\epsilon} = \inf_{\substack{v \in W^{m,2}(\Omega_{\epsilon}) \setminus \{0\}\\v=0 \text{ on } K_{\epsilon}}} \frac{Q_{\Omega_{\epsilon}}(v)}{\|v\|_{L^{2}(\Omega_{\epsilon})}^{2}}$$

Then we have $\lim_{\epsilon \to 0} |\Omega_{\epsilon} \setminus K_{\epsilon}| = 0$ and condition (**C**) is satisfied. Hence $H_{W^{m,2}(\Omega_{\epsilon})}^{-1} \xrightarrow{\mathcal{C}} H_{W^{m,2}(\Omega)}^{-1}$.

Remark 3. i) In the particular, but very interesting situation of an exterior perturbation of the domain, that is $\Omega \subset \Omega_{\epsilon}$, we may choose $K_{\epsilon} \equiv \Omega$ in Theorem 1.

ii) Note that τ_{ϵ} can be thought as the first eigenvalue of the operator (6) on $\Omega_{\epsilon} \setminus \bar{K}_{\epsilon}$ subject to Neumann boundary conditions on $\partial \Omega_{\epsilon}$ and Dirichlet boundary conditions on ∂K_{ϵ} .

iii) As it is shown in [2], if the boundary of Ω_{ϵ} can be described locally as the graph of a function F_{ϵ} defined over $\partial \Omega$ then condition ii) is satisfied. The function F_{ϵ} maybe very oscillating but still $\tau_{\epsilon} \stackrel{\epsilon \to 0}{\longrightarrow} \infty$.

4. Bi-harmonic operator and intermediate boundary conditions

In this section, we consider intermediate boundary conditions for the 4th-order bi-harmonic operator. That is, we consider $A_{\alpha\beta} = \delta_{\alpha\beta}$, so that the bilinear form looks now:

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$$Q_{\Omega}(u,v) = \sum_{i,j=1}^{N} \int_{\Omega} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} \, dx + \int_{\Omega} uv \, dx \tag{8}$$

and $V(\Omega) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. We will denote by H_{Ω}^I the self-adjoint operator corresponding to this bilinear form in $V(\Omega)$, as constructed in Section 1. If the domain Ω is smooth, one can easily see via integration by parts that this operator corresponds to the differential operator $Lu = \Delta^2 u + u$, with boundary conditions:

$$u = 0$$
, and $\Delta u - K \frac{\partial u}{\partial v} = 0$, on $\partial \Omega$, (9)

where K denotes the mean curvature of $\partial \Omega$, i.e. the sum of the principal curvatures. The behavior of the spectrum of this operator under perturbations of the domain is known to be somehow tricky and the famous Babuška paradox is related to it, see, e.g., [13] for a recent discussion.

We want to consider the behavior of the spectrum of H_{Ω}^{I} , given by (7), when the boundary of a fixed domain Ω is perturbed in an oscillatory way. To simplify the exposition, let us assume that our domain $\Omega \subset \mathbb{R}^N$ is such that $\partial \Omega \cap \{x_N = 0\} = W$ where $W \subset \mathbb{R}^{N-1}$ is a bounded and smooth domain in \mathbb{R}^{N-1} . Moreover, we will assume that $(W \times (-1, 1)) \cap \Omega = 0$ $W \times (-1, 0)$. The perturbed domain Ω_{ϵ} will differ from Ω only in the part of the boundary given by W. As a matter of fact, we will assume that there exists a function $g_{\epsilon}: W \to (-\frac{1}{2}, \frac{1}{2})$ so that if we set $\omega_{\epsilon} = \{(\bar{x}, x_N): \bar{x} \in W, -1 < x_N < g_{\epsilon}(\bar{x})\}$, then $\Omega_{\epsilon} = (\Omega \setminus (W \times (-1, 0))) \cup \omega_{\epsilon}$. Moreover, we will assume that the one-parameter family of functions g_{ϵ} are of the type: $g_{\epsilon}(\bar{x}) = \epsilon^{\alpha} g(\bar{x}/\epsilon)$, where the fixed function $g: \mathbb{R}^{N-1} \to (-\frac{1}{2}, \frac{1}{2})$ is Y-periodic, Y is the unit cell $Y = (-\frac{1}{2}, \frac{1}{2})^{N-1}$ and $\alpha > 0.$

Notice that as long as $\alpha > 0$, we have that $g_{\epsilon} \to 0$ uniformly in \mathbb{R}^{N-1} . Actually, it is possible to see that g_{ϵ} is bounded in C^{α} and $g_{\epsilon} \to 0$ in C^{α^-} , where α^- is any number smaller than α . For instance, if $\alpha > 2$ then $g_{\epsilon} \to 0$ in C^2 and therefore, since the curvature appears in the boundary conditions, see (9), we expect that the limit problem will also have the boundary conditions (9). In particular $K \equiv 0$ in W. But it is not clear at all what should happen when $\alpha \leq 2$. For instance if $\alpha > 1$ but near 1, we have the C¹ convergence of the boundary, but this is not enough to guarantee the convergence of the curvature. On the other hand, if $\alpha > 0$ but near 0, the oscillations of the boundary are very extreme, in the sense that the amplitude is much larger than the period and, since our solutions satisfy u = 0 at the boundary (since $V(\Omega) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, it seems plausible that the limiting boundary condition in W should be Dirichlet (that is $u = \frac{\partial u}{\partial n} = 0$). This indicates that there must exist a critical value that marks a threshold on the behavior. As a matter of fact, we will see that independently of the dimension *N* of the domain, the critical value is $\alpha = 3/2$.

To clarify the notation, we will also denote by H_{Ω}^{D} the self-adjoint operator corresponding to the bilinear form (8) with Dirichlet boundary conditions (that is $V(\Omega) = W_0^{2,2}(\Omega)$). We have the following result.

Theorem 4.1. With the notations above, we have the following trichotomy:

- i) If $\alpha > 3/2$, then $H^I_{\Omega_c} \xrightarrow{\mathcal{C}} H^I_{\Omega}$. ii) If $0 < \alpha < 3/2$, then $H_{\Omega_{\epsilon}}^{I} \xrightarrow{C} H_{\Omega}^{D}$.
- iii) If $\alpha = 3/2$, then $H_{\Omega_{\epsilon}}^{I} \xrightarrow{\mathcal{C}} \hat{H}_{\Omega}$, where \hat{H}_{Ω} is the bi-harmonic operator in Ω with the following boundary conditions in W: u = $0, \Delta u - \gamma \frac{\partial u}{\partial v} = 0$, where the factor γ is given as:

$$\gamma = \int_{Y} \frac{\partial}{\partial \nu} (\Delta_Y \hat{\nu} + \Delta \hat{\nu}) \, \mathrm{d}y,$$

where Δ_Y is the Laplace operator in the Y-variables and the function \hat{v} is Y-periodic and satisfies the following microscopic problem:

$$\begin{cases} \Delta^{2} \hat{v} = 0, & \text{in } Y \times (-\infty, 0), \\ \hat{v}(y, 0) = g(y), & \text{on } Y, \\ \frac{\partial^{2} \hat{v}}{\partial v^{2}}(y, 0) = 0, & \text{on } Y. \end{cases}$$
(10)

Idea of the proof. To simplify, let us consider the case where Ω is the parallelepiped $\Omega = W \times (-2, 0)$ with $W = (-1, 1)^{N-1}$ and the function g > 0, so that $\Omega \subset \Omega_{\epsilon}$.

i) We will show that condition (C) from Definition 2.3 holds. In particular, we need to construct an appropriate transformation $T_{\epsilon}: V(\Omega) \to V(\Omega_{\epsilon})$. Notice that the space $V = W^{2,2} \cap W_0^{1,2}$ makes the construction of these maps not

so trivial. We cannot directly use extension and restriction operators. Hence, we will define a transformation via the so-called pullback transformation associated with the map $\Phi_{\epsilon}: \Omega_{\epsilon} \to \Omega$, given as $\Phi_{\epsilon}(\bar{x}, x_N) = (\bar{x}, x_N - h_{\epsilon}(\bar{x}, x_N))$:

$$h_{\epsilon}(\bar{x}, x_N) = \begin{cases} 0, & -1 < x_N \leqslant -\epsilon, \\ g_{\epsilon}(\bar{x})(\frac{x_N + \epsilon}{g_{\epsilon}(\bar{x}) + \epsilon})^3, & -\epsilon < x_N < g_{\epsilon}(\bar{x}). \end{cases}$$

The map T_{ϵ} is defined as $T_{\epsilon}(u) = u \circ \Phi_{\epsilon}$.

A delicate analysis of this map reveals that if $\alpha > 3/2$, then T_{ϵ} satisfies i), ii) and iii) from (C2). This opens the possibility to show that condition (C) actually holds true for the case $\alpha > 3/2$.

ii) In this case, the operator T_{ϵ} defined above does not satisfy (C2) ii). As a matter of fact, it is possible to show that, for any family of functions $u_{\epsilon} \in W^{2,2}(\Omega_{\epsilon}) \cap W_0^{1,2}(\Omega_{\epsilon})$ such that $||u_{\epsilon}||_{W^{2,2}(\Omega_{\epsilon})} \leq C$ and such that possibly passing to a subsequence, $u_{\epsilon|\Omega} \to u$ weakly in $W^{2,2}(\Omega)$ as $\epsilon \to 0$, then $u \in W_0^{2,2}(\Omega)$. By exploiting this fact, we may show that

 $H_{\Omega_{c}}^{I} \xrightarrow{\mathcal{C}} H_{\Omega}^{D}$ if $0 < \alpha < 3/2$.

iii) By large, this case is the most delicate. We will be able to obtain the limit via an application of the unfolding operator method in a similar spirit as the one from [12]. We refer the reader to [4] for details. \Box

Remark 4. i) A particular case of ii) where the convergence to Dirichlet boundary condition is shown for a 2-D domain under polygonal approximations is obtained originally in [14]. See also, [15, Section 18.3, Volume II].

ii) The exponent $\alpha = 3/2$ is independent of the dimension. Moreover, this threshold exponent coincides with the one for the Stokes operator, as in [12].

iii) It can be seen that, for cases i) and ii), we do not really need a periodic structure in the perturbation. Actually, any perturbation with g bounded in $C^2(\mathbb{R}^{N-1},\mathbb{R})$ will also satisfy i). For case ii), any perturbation with g bounded in $C^{2}(\mathbb{R}^{N-1},\mathbb{R})$, which is essentially not constant, will also satisfy ii).

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