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Partial differential equations

Finite time singularity in a free boundary problem modeling MEMS

Singularité en temps fini pour un modèle de microsystème électromécanique à frontière libre

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ABSTRACT

The occurrence of a finite time singularity is shown for a free boundary problem modeling microelectromechanical systems (MEMS) when the applied voltage exceeds some value. The model involves a singular nonlocal reaction term and a nonlinear curvature term taking into account large deformations.

RÉSUMÉ

L'apparition d'une singularité en temps fini est établie pour un problème à frontière libre décrivant l'évolution spatio-temporelle d'un microsystème électromécanique lorsque la tension appliquée est suffisamment élevée. Le modèle inclut un terme de réaction singulier et un terme non linéaire de courbure, prenant en compte les grandes déformations.
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Version française abrégée

Un transducteur électrostatique ou capacitif est un microsystème électromécanique constitué de deux électrodes, l'une étant rigide et fixe et l'autre étant assimilée à une membrane élastique et fixée à ses deux extrémités. L'application d'une tension électrique entre les électrodes engendre une force électrostatique qui induit une déformation de la membrane élastique et convertit ainsi l'énergie électrique en énergie mécanique. La dynamique du dispositif est alors décrite par la déformation verticale u de la membrane élastique et le potentiel électrostatique entre les deux électrodes ψ. Si on suppose qu'il n'y a pas de variation dans la direction horizontale transverse, un changement d'échelle permet de supposer que l'électrode fixe est située en $z=-1$ et que la déformation de la membrane $u=u(t,x)\in(-1,\infty)$ à l'instant $t>0$ et à la position $x\in I:=(-1,1)$ vérifie l'équation (1) avec les conditions limites (2) et initiale (3). Le potentiel électrostatique est solution de l'équation de Laplace renormalisée (4) dans la région située entre les deux électrodes :

$$\Omega\big(u(t)\big):=\big\{(x,z)\in I\times(-1,\infty):\ -1<z<u(t,x)\big\}$$

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avec les conditions limites (5), le paramètre $\varepsilon > 0$ désignant le rapport entre les dimensions verticale et horizontales du dispositif et le paramètre $\lambda > 0$ étant proportionnel au carré de la tension électrique appliquée. L'évolution spatio-temporelle de (u, ψ) est donc régie par le couplage d'une équation parabolique quasilinéaire pour la déformation u et d'une équation elliptique pour le potentiel électrostatique ψ. Cette dernière est posée dans le domaine dépendant du temps $\Omega(u(t))$ et n'est bien définie que tant que la membrane élastique ne touche pas l'électrode rigide, c'est-à-dire tant que $u > -1$. Cette situation (collage) peut effectivement se produire et dépend fortement de la valeur de λ. Ce phénomène doit être contrôlé afin de garantir un fonctionnement optimal du dispositif.

La relation entre le terme source dans (1) et la déformation u est clairement implicite, non locale et non linéaire, et c'est l'une des difficultés rencontrées dans l'analyse mathématique du système (1)–(5), l'autre étant le comportement singulier du terme source dans (1) lorsque $u \to -1$.

Nous avons récemment montré dans [5] le caractère localement bien posé de (1)–(5) pour des données initiales suffisamment régulières. Plus précisément, étant donnés $q > 2$ et une condition initiale $u^0 \in W_q^2(I)$ vérifiant $u^0(\pm 1) = 0$ et $0 \geqslant u^0(x) > -1$, $x \in I$, il existe une unique solution maximale (u, ψ) de (1)–(5) définie pour $t \in [0, T_m^\varepsilon)$ et telle que $0 \geqslant u(t, x) > -1$, $(t, x) \in [0, T_m^\varepsilon) \times I$ (Théorème 1). Nous avons, de plus, établi que cette solution existe globalement pourvu que λ et $\|u^0\|_{W_q^2(I)}$ soient suffisamment petits. Dans ce cas, il existe $\kappa \in (0, 1)$ tel que $u(t, x) > -1 + \kappa$ pour $(t, x) \in [0, \infty) \times I$ (Théorème 2).

Dans cette note, nous étudions le comportement des solutions de (1)–(5) lorsque λ est suffisamment grand et montrons la formation d'une singularité en temps fini (Théorème 3). Plus précisément, $T_m^\varepsilon < \infty$ dès que $\lambda > 1/\varepsilon$. D'une part, ce résultat complète un résultat de non-existence de solutions stationnaires pour les grandes valeurs de λ obtenu dans notre article précédent [5]. D'autre part, une conséquence du résultat d'existence local [5, Theorem 1.1 (ii)] est que le caractère fini de T_m^ε peut correspondre à deux types de singularité lorsque $t \to T_m^\varepsilon$: le phenomène de collage de la membrane sur l'électrode rigide, c'est-à-dire $\min_{[-1,1]} u(t) \longrightarrow -1$ lorsque $t \to T_m^\varepsilon$, ou l'explosion au temps T_m^ε de la norme $W_q^2(I)$ de u.

1. Introduction

An idealized electrostatically actuated microelectromechanical system (MEMS) consists of a fixed horizontal ground plate held at zero potential, above which an elastic membrane held at potential V is suspended. A Coulomb force is generated by the potential difference across the device and results in a deformation of the membrane, thereby converting electrostatic energy into mechanical energy, see [1,6,10] for a more detailed account and further references. After a suitable scaling and assuming homogeneity in transversal horizontal direction, the ground plate is assumed to be located at $z = -1$ and the membrane displacement $u = u(t, x) \in (-1, \infty)$ with $t > 0$ and $x \in I := (-1, 1)$ evolves according to:

$$\partial_t u - \partial_x \left(\frac{\partial_x u}{\sqrt{1 + \varepsilon^2 (\partial_x u)^2}} \right) = -\lambda \left(\varepsilon^2 \left| \partial_x \psi\big(t, x, u(t, x)\big) \right|^2 + \left| \partial_z \psi\big(t, x, u(t, x)\big) \right|^2 \right), \tag{1}$$

for $t > 0$ and $x \in I$ with boundary conditions:

$$u(t, \pm 1) = 0, \quad t > 0, \tag{2}$$

and the initial condition:

$$u(0, x) = u^0(x), \quad x \in I. \tag{3}$$

The electrostatic potential $\psi = \psi(t, x, z)$ satisfies a rescaled Laplace equation in the region:

$$\Omega\big(u(t)\big) := \big\{(x, z) \in I \times (-1, \infty): \ -1 < z < u(t, x)\big\}$$

between the plate and the membrane, which reads:

$$\varepsilon^2 \partial_x^2 \psi + \partial_z^2 \psi = 0, \quad (x, z) \in \Omega\big(u(t)\big), \quad t > 0, \tag{4}$$

$$\psi(t, x, z) = \frac{1 + z}{1 + u(t, x)}, \quad (x, z) \in \partial\Omega\big(u(t)\big), \quad t > 0, \tag{5}$$

where $\varepsilon > 0$ denotes the aspect ratio of the device and $\lambda > 0$ is proportional to the square of the applied voltage. The dynamics of (u, ψ) is thus given by the coupling of a quasilinear parabolic equation for u and an elliptic equation in a moving domain for ψ, the latter being only well defined as long as the membrane does not touch down on the ground plate, that is, u does not reach the value -1. To guarantee optimal operating conditions of the device, this touchdown phenomenon has to be controlled and its occurrence is obviously related to the value of λ.

The main difficulty to be overcome in the analysis of (1)–(5) is the nonlocal and nonlinear implicit dependence on u of the right-hand side of (1), which is also singular if u approaches -1. Except for the singularity, these features disappear when setting $\varepsilon = 0$ in (1)–(5), a commonly made assumption that reduces (1)–(5) to a singular semilinear reaction–diffusion equation (for a quasilinear variant thereof, see [2]). This so-called small aspect ratio model has received considerable attention in recent years, see [6,10] and the references therein. In this simplified situation, it has been established that touchdown does not take place if λ is below a certain threshold value $\lambda_* > 0$, but occurs if λ exceeds this value [6–8]. Another simplification is the small deformation approximation, which reduces the second-order differential term on the right-hand side of (1) to $\partial_x^2 u$ and has been studied in [3,4,9].

We have recently investigated the well-posedness of the evolution problem (1)–(5) and established the following result [5].

Theorem 1 *(Local Well-Posedness). Let $q \in (2,\infty)$, $\varepsilon > 0$, $\lambda > 0$, and consider an initial value*:

$$u^0 \in W_q^2(I) \quad \text{such that} \quad u^0(\pm 1) = 0 \quad \text{and} \quad 0 \geqslant u^0(x) > -1 \quad \text{for} \quad x \in I. \tag{6}$$

Then there is a unique solution (u,ψ) to (1)–(5) on the maximal interval of existence $[0, T_m^\varepsilon)$ in the sense that:

$$u \in C^1\big(\big[0, T_m^\varepsilon\big), L_q(I)\big) \cap C\big(\big[0, T_m^\varepsilon\big), W_q^2(I)\big)$$

satisfies (1)–(3) together with:

$$0 \geqslant u(t,x) > -1, \quad (t,x) \in \big[0, T_m^\varepsilon\big) \times I, \tag{7}$$

and $\psi(t) \in W_2^2(\Omega(u(t)))$ solves (4)–(5) for each $t \in [0, T_m^\varepsilon)$.

We have also shown in [5] that, if λ and u^0 are sufficiently small, the solution (u,ψ) to (1)–(5) exists for all times (i.e. $T_m^\varepsilon = \infty$) and touchdown does not take place, not even in infinite time.

Theorem 2 *(Global Existence). Let $q \in (2,\infty)$, $\varepsilon > 0$, and consider an initial value u^0 satisfying (6). Given $\kappa \in (0,1)$, there are $\lambda_*(\kappa) > 0$ and $r(\kappa) > 0$ such that, if $\lambda \in (0, \lambda_*(\kappa))$ and $\|u^0\|_{W_q^2(I)} \leqslant r(\kappa)$, the solution (u,ψ) to (1)–(5) exists for all times and $u(t,x) \geqslant -1+\kappa$ for $(t,x) \in [0,\infty) \times I$.*

On the other hand, we have been able to prove that no stationary solution to (1)–(5) exists provided that λ is sufficiently large. However, whether or not T_m^ε is finite in this case has been left as an open question. The purpose of this note is to show that – as expected on physical grounds – T_m^ε is indeed finite for λ sufficiently large.

Theorem 3 *(Finite-time singularity). Let $q \in (2,\infty)$, $\varepsilon > 0$, and consider an initial value u^0 satisfying (6). If $\lambda > 1/\varepsilon$ and (u,ψ) denotes the maximal solution to (1)–(5) defined on $[0, T_m^\varepsilon)$, then $T_m^\varepsilon < \infty$.*

The criterion $\lambda > 1/\varepsilon$ is likely to be far from optimal. As we shall see below, improving it would require to have a better control on $\partial_x u(\pm 1)$. The proof of Theorem 3 relies on the derivation of a chain of estimates that allows us to obtain a lower bound on the L_1-norm of the right-hand side of (1), depending only on u. The lower bound thus obtained is in fact the mean value of a convex function of u, and we may then end the proof with the help of Jensen's inequality, an argument that has already been used for the small aspect ratio model, see [7,8].

We shall point out that, in contrast to the small aspect ratio model, the finiteness of T_m^ε does not guarantee that the touchdown phenomenon really takes place as $t \to T_m^\varepsilon$. Indeed, according to [5, Theorem 1.1 (ii)], the finiteness of T_m^ε implies that $\min_{[-1,1]} u(t) \longrightarrow -1$ or $\|u(t)\|_{W_q^2(I)} \longrightarrow \infty$ as $t \to T_m^\varepsilon$. While the former corresponds to the touchdown behaviour, the latter is more likely to be interpreted as the membrane being no longer the graph of a function at time T_m^ε.

2. Proof of Theorem 3

Let $q \in (2,\infty)$, $\varepsilon > 0$, $\lambda > 0$ and consider an initial value u^0 satisfying (6). We denote the maximal solution to (1)–(5) defined on $[0, T_m^\varepsilon)$ by (u,ψ). Differentiating the boundary conditions (5), we readily obtain:

$$\partial_x \psi(t,x,-1) = \partial_x \psi\big(t,x,u(t,x)\big) + \partial_x u(t,x) \partial_z \psi\big(t,x,u(t,x)\big) = 0, \quad (t,x) \in \big(0, T_m^\varepsilon\big) \times I, \tag{8}$$

and

$$\partial_z \psi(t, \pm 1, z) = 1, \quad (t,z) \in \big(0, T_m^\varepsilon\big) \times (-1,0). \tag{9}$$

Additional information on the boundary behaviour of ψ is provided by the next lemma.

Lemma 4. *For $t \in (0, T_m^\varepsilon)$,*

$$1 + z \leqslant \psi(t,x,z) \leqslant 1, \qquad (x,z) \in \Omega\big(u(t)\big), \tag{10}$$

$$\pm \partial_x \psi(t, \pm 1, z) \leqslant 0, \qquad z \in (-1,0). \tag{11}$$

Proof. Fix $t \in (0, T_m^\varepsilon)$. The upper bound in (10) readily follows from the maximum principle. Next, the function σ, defined by $\sigma(x,z) = 1+z$, obviously satisfies $\varepsilon^2 \partial_x^2 \sigma + \partial_z^2 \sigma = 0$ in $\Omega(u(t))$ as well as:

$$\sigma(\pm 1, z) = 1 + z = \psi(t, \pm 1, z), \quad z \in (-1, 0),$$
$$\sigma(x, -1) = 0 = \psi(t, x, -1), \quad x \in (-1, 1).$$

Owing to the non-positivity (7) of $u(t)$, it also satisfies:

$$\sigma\big(x, u(t,x)\big) = 1 + u(t,x) \leqslant 1 = \psi\big(t, x, u(t,x)\big), \quad x \in (-1, 1),$$

and we infer from the comparison principle that $\psi(t,x,z) \geqslant \sigma(x,z)$ for $(x,z) \in \Omega(u(t))$. It then follows from (10) that $\psi(t,x,z) \geqslant 1 + z = \psi(t, \pm 1, z)$ for $(x,z) \in \Omega(u(t))$, which readily implies (11). □

To simplify notations, we set:

$$\gamma_m(t,x) := \partial_z \psi\big(t, x, u(t,x)\big), \qquad \gamma_g(t,x) := \partial_z \psi(t, x, -1), \quad (t,x) \in \big(0, T_m^\varepsilon\big) \times (-1, 1), \tag{12}$$

and first derive an upper bound of the L_1-norm of the right-hand side of (1), observing that, due to (8), it also reads:

$$-\lambda\big(\varepsilon^2 \big|\partial_x \psi\big(t, x, u(t,x)\big)\big|^2 + \big|\partial_z \psi\big(t, x, u(t,x)\big)\big|^2\big) = -\lambda\big(1 + \varepsilon^2 \big(\partial_x u(t,x)\big)^2\big)\gamma_m(t,x)^2.$$

Lemma 5. *For* $t \in (0, T_m^\varepsilon)$,

$$\int_{-1}^{1} \big(1 + \varepsilon^2 \big(\partial_x u(t,x)\big)^2\big)\gamma_m(t,x)^2 \, \mathrm{d}x \geqslant 2 \int_{-1}^{1} \big(1 + \varepsilon^2 \big(\partial_x u(t,x)\big)^2\big)\gamma_m(t,x) \, \mathrm{d}x - 2. \tag{13}$$

Proof. Fix $t \in (0, T_m^\varepsilon)$. We multiply (4) by $\partial_z \psi(t) - 1$ and integrate over $\Omega(u(t))$. Using (8), (9), and Green's formula we obtain:

$$0 = -\varepsilon^2 \int_{\Omega(u)} \partial_x \partial_z \psi \; \partial_x \psi \, \mathrm{d}(x,z) + \varepsilon^2 \int_{-1}^{1} (\partial_x u)^2 \; \gamma_m(\gamma_m - 1) \, \mathrm{d}x$$
$$- \frac{1}{2} \int_{-1}^{1} \big(\gamma_g^2 - 2\gamma_g\big) \, \mathrm{d}x + \frac{1}{2} \int_{-1}^{1} \big(\gamma_m^2 - 2\gamma_m\big) \, \mathrm{d}x.$$

Since

$$\int_{\Omega(u)} \partial_x \partial_z \psi \; \partial_x \psi \, \mathrm{d}(x,z) = \frac{1}{2} \int_{-1}^{1} (\partial_x u)^2 \gamma_m^2 \, \mathrm{d}x$$

by (8) and since $\gamma_g^2 - 2\gamma_g \geqslant -1$, we end up with (13). □

We again use (4) to obtain a lower bound for the boundary integral of the right-hand side of (13), which depends on the Dirichlet energy of ψ.

Lemma 6. *For* $t \in (0, T_m^\varepsilon)$,

$$\int_{-1}^{1} \big(1 + \varepsilon^2 \big(\partial_x u(t,x)\big)^2\big)\gamma_m(t,x) \, \mathrm{d}x \geqslant \int_{\Omega(u(t))} \big(\varepsilon^2 \big|\partial_x \psi(t,x,z)\big|^2 + \big|\partial_z \psi(t,x,z)\big|^2\big) \, \mathrm{d}(x,z). \tag{14}$$

Proof. Fix $t \in (0, T_m^\varepsilon)$. We multiply (4) by $\psi(t)$ and integrate over $\Omega(u(t))$. Using (5), (8), and Green's formula we obtain:

$$0 = -\int_{\Omega(u(t))} \big(\varepsilon^2 \big|\partial_x \psi(t,x,z)\big|^2 + \big|\partial_z \psi(t,x,z)\big|^2\big) \, \mathrm{d}(x,z) + \varepsilon^2 \int_{-1}^{0} (1+z)\partial_x \psi(t, 1, z) \, \mathrm{d}z$$
$$- \varepsilon^2 \int_{-1}^{0} (1+z)\partial_x \psi(t, -1, z) \, \mathrm{d}z + \varepsilon^2 \int_{-1}^{1} \big(\partial_x u(t,x)\big)^2 \; \gamma_m(t,x) \, \mathrm{d}x + \int_{-1}^{1} \gamma_m(t,x) \, \mathrm{d}x.$$

Owing to (11), the second and third terms of the right-hand side of the above equality are non-positive, whence (14). □

We finally argue as in [4, Lemma 9] to establish a connection between the Dirichlet energy of ψ and u.

Lemma 7. *For $t \in (0, T_m^\varepsilon)$,*

$$\int_{\Omega(u(t))} \left(\varepsilon^2 \left|\partial_x \psi(t,x,z)\right|^2 + \left|\partial_z \psi(t,x,z)\right|^2\right) \mathrm{d}(x,z) \geqslant \int_{-1}^{1} \frac{\mathrm{d}x}{1+u(t,x)}. \tag{15}$$

Proof. Let $t \in (0, T_m^\varepsilon)$ and $x \in (-1,1)$. We deduce from (5) and the Cauchy–Schwarz inequality that:

$$\begin{aligned} \frac{1}{1+u(t,x)} &= \frac{(\psi(t,x,u(t,x)) - \psi(t,x,-1))^2}{1+u(t,x)} = \frac{1}{1+u(t,x)} \left(\int_{-1}^{u(t,x)} \partial_z \psi(t,x,z)\,\mathrm{d}z \right)^2 \\ &\leqslant \int_{-1}^{u(t,x)} \left(\partial_z \psi(t,x,z)\right)^2 \mathrm{d}z. \end{aligned} \tag{16}$$

Integrating the above inequality with respect to $x \in (-1,1)$ readily gives (15). □

Remark 8. Observe that (16) provides a quantitative estimate on the singularity of $\partial_z \psi$ generated by u when touchdown occurs.

Combining the three lemmas above with Jensen's inequality gives the following estimate.

Proposition 9. *For $t \in (0, T_m^\varepsilon)$,*

$$\int_{-1}^{1} \left(1 + \varepsilon^2 \left(\partial_x u(t,x)\right)^2\right) \gamma_m(t,x)^2 \,\mathrm{d}x \geqslant 4\varphi\left(\frac{1}{2} \int_{-1}^{1} u(t,x)\,\mathrm{d}x \right) - 2, \tag{17}$$

where $\varphi(r) := 1/(1+r)$, $r \in (-1, \infty)$.

Proof. Fix $t \in (0, T_m^\varepsilon)$. We infer from Lemma 5, Lemma 6, and Lemma 7 that:

$$\int_{-1}^{1} \left(1 + \varepsilon^2 \left(\partial_x u(t,x)\right)^2\right) \gamma_m(t,x)^2 \,\mathrm{d}x \geqslant 2 \int_{-1}^{1} \varphi\big(u(t,x)\big)\,\mathrm{d}x - 2.$$

To complete the proof, we argue as in [7,8] and use the convexity of φ and Jensen's inequality to obtain (17). □

Proof of Theorem 3. Introducing:

$$E(t) := -\frac{1}{2} \int_{-1}^{1} u(t,x)\,\mathrm{d}x, \quad t \in \left[0, T_m^\varepsilon\right),$$

the bounds (7) ensure that:

$$0 \leqslant E(t) < 1, \quad t \in \left[0, T_m^\varepsilon\right). \tag{18}$$

It follows from (1), (8), and Proposition 9 that:

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t}(t) &= -\frac{1}{2} \left[\frac{\partial_x u(t,x)}{\sqrt{1 + \varepsilon^2 (\partial_x u(t,x))^2}} \right]_{x=-1}^{x=1} + \frac{\lambda}{2} \int_{-1}^{1} \left(1 + \varepsilon^2 \left(\partial_x u(t,x)\right)^2\right) \gamma_m(t,x)^2 \,\mathrm{d}x \\ &\geqslant F_\lambda(E) := 2\lambda \varphi(-E) - \lambda - \frac{1}{\varepsilon}. \end{aligned} \tag{19}$$

If $\lambda > 1/\varepsilon$, we note that $F_\lambda(0) > 0$ and thus $F_\lambda(r) \geqslant F_\lambda(0) > 0$ for $r \in [0,1)$ due to the monotonicity of F_λ. Since $E(0) \geqslant 0$ by (18), it follows from (19) and the properties of F_λ that $t \mapsto E(t)$ is increasing on $[0, T_m^\varepsilon)$. Consequently,

$$\frac{\mathrm{d}E}{\mathrm{d}t}(t) \geqslant F_\lambda\big(E(0)\big) \geqslant F_\lambda(0), \quad t \in \big[0, T_m^\varepsilon\big).$$

Integrating the previous inequality with respect to time and using (18), we end up with the inequality $1 \geqslant E(0) + F_\lambda(0) T_m^\varepsilon$, which provides the claimed finiteness of T_m^ε. □

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