

# Contents lists available at ScienceDirect

# C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com



# Remarks on compact shrinking Ricci solitons of dimension four $\overset{\scriptscriptstyle \star}{}$



Li Ma<sup>a,b</sup>

<sup>a</sup> Department of Mathematics, Henan Normal University, Xinxiang 453007, China
 <sup>b</sup> Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

#### ARTICLE INFO

Article history: Received 17 December 2012 Accepted after revision 9 October 2013 Available online 30 October 2013

Presented by the Editorial Board

### ABSTRACT

In this paper, we study the topological restriction of gradient shrinking Ricci solitons (M, g) of dimension 4. Let *s* be the scalar curvature of the metric *g*. Then we have:

$$\int_{M} s \, \mathrm{d} v_g = 4\rho \, \mathrm{vol}(M).$$

where  $\rho > 0$  is the shrinking constant and vol(*M*) is the volume of (*M*, *g*). We also have two kinds of topology results. (1) If we assume that:

$$\int_{M} s^2 \leqslant 24\rho^2 \operatorname{vol}(M),$$

then

$$2\chi(M) \pm 3\tau(M) \ge 0.$$

(2) If (M, g) is a natural oriented Kähler surface, then we have:

$$2\chi(M) + 3\tau(M) = \frac{\rho^2 \operatorname{vol}(M)}{2\pi^2}.$$

Actually, we shall show that the assumption in (1) above is equivalent to the fact that:

$$\int_{M} \sigma_2 \left( Rc - \frac{s}{6} g \right) \ge 0.$$

Here  $\sigma_2(A) := \sigma_2(g)$  is the 2nd symmetric function of the eigenvalues of the matrix  $A := Rc - \frac{5}{6}g$ .

© 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Nous étudions dans cette Note la restriction topologique des solitons de Ricci (M, g) contractant le gradient, de dimension 4. Soit *s* la courbure scalaire de la métrique g, alors

E-mail address: lma@tsinghua.edu.cn.

1631-073X/\$ – see front matter © 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved. http://dx.doi.org/10.1016/j.crma.2013.10.006





<sup>\*</sup> The research is partially supported by the National Natural Science Foundation of China No. 11271111 and SRFDP 20090002110019. The version here is a revised version after a suggestion from Prof. C. LeBrun during Muenster conference in August of 2006.

on a :

$$\int_{M} s \, \mathrm{d} v_{\mathrm{g}} = 4\rho \, \mathrm{vol}(M),$$

où  $\rho > 0$  est la constante de contraction et vol(*M*) le volume de (*M*, *g*). Nous obtenons également deux types de résultats topologiques. (1) En supposant :

$$\int_{M} s^2 \leqslant 24\rho^2 \operatorname{vol}(M),$$

alors

 $2\chi(M) \pm 3\tau(M) \ge 0.$ 

(2) Si (M, g) est une surface de Kähler naturellement orientée, alors on a :

$$2\chi(M) + 3\tau(M) = \frac{\rho^2 \operatorname{vol}(M)}{2\pi^2}$$

En fait, nous montrons que l'hypothèse de (1) ci-dessus est équivalente à :

$$\int_{M} \sigma_2\left(Rc-\frac{s}{6}g\right) \ge 0,$$

avec  $\sigma_2(Rc - \frac{s}{6}g) := \sigma_2(g)$  la seconde fonction symétrique des valeurs propres de la matrice  $Rc - \frac{s}{6}g$ .

© 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

#### 1. Introduction

It has been shown by G. Perelman [22] that there is no non-trivial compact steady and expanding Ricci soliton (and they are all Einstein). Roughly speaking, Ricci solitons are self-similar solutions to the Ricci flow. Even in dimension three, the global Ricci flow on complete non-compact Riemannian manifolds is not well understood [20]. Special examples of Ricci solitons are Einstein metrics, which are fixed points of the normalized Ricci flow. In dimensions two and three, there is no compact non-trivial Ricci soliton, and we have non-compact Ricci solitons like Bryant soliton in dimension three and the cigar soliton of R. Hamilton (also called Witten black hole) in dimension two, see [5]. We point out here that the dimension-four case is special by the fact that we do have compact non-trivial Ricci solitons like Cao and Koiso solitons, see [6]. Using Corollary 1.2 in [23], we know that  $CP^2 \sharp 2C\overline{P^2}$  admits a non-Einstein Kähler–Ricci soliton metric. One of the most important topics in Ricci flow is to classify the compact Ricci solitons in dimension 4. Just like in the Einstein metric case, we can expect that there are some topological restrictions to Ricci solitons and there is some type of Hitchin–Thorpe inequality for Ricci solitons. Our aim here is to show that this is true for some cases. We write by *s* the scalar curvature of the Riemannian metric *g*. Our main result is below.

**Theorem 1.** Let (M, g) be a gradient shrinking Ricci soliton of dimension 4 with a shrinking constant  $\rho > 0$ . Then we have:

$$\int_{M} s \, \mathrm{d} v_g = 4\rho \, \mathrm{vol}(M).$$

We have the following two topology results. (1) There holds:

$$2\chi(M) \pm 3\tau(M) = -\frac{1}{48\pi^2} \int_M s^2 + \frac{\rho^2}{2\pi^2} \operatorname{vol}(M) + \frac{1}{2\pi^2} \int_M |W_{\pm}|^2 \, \mathrm{d}\nu_g \tag{1}$$

where  $W = W_+ + W_-$  is the Weyl tensor of the metric g.

(2) If (M, g) is a naturally oriented Kähler surface, then we have:

$$2\chi(M) + 3\tau(M) = \frac{\rho^2 \operatorname{vol}(M)}{2\pi^2}$$

Here the word "naturally oriented" means that the Kähler form is self-dual as in [4].

This result is a generalization of the Hitchin–Thorpe inequality (see A. Besse's book [4]) for Einstein metrics.

**Corollary 2.** Given (M, g) the gradient shrinking Ricci soliton of dimension 4 with a shrinking constant  $\rho > 0$  as above. If we further assume that:

$$\int_{M} s^2 \leqslant 24\rho^2 \operatorname{vol}(M),\tag{2}$$

then

$$2\chi(M) \pm 3\tau(M) \ge \frac{1}{2\pi^2} \int_{M} |W_{\pm}|^2 \ge 0.$$

We basically show that the condition (2) is equivalent to the fact that:

$$\int_{M} \sigma_2\left(Rc-\frac{s}{6}g\right) \ge 0.$$

Here  $\sigma_2(A)$  is the 2nd elementary symmetric function of the eigenvalues of the matrix  $A := Rc - \frac{5}{6}g$ . Then the result follows from the Chern-Gauss-Bonnet Theorem. There is a large room to be done for more results. For example, it is expected that there are some close relationships between Einstein metrics and Ricci solitons on compact Kähler manifolds. One also likes to see some applications of Ricci-Kähler solitons to algebraic geometry [24] or conformal field theory. As suggested by Prof. LeBrun, using the Seiberg–Witten invariant method of C. LeBrun [17], one may sharpen the inequality above for Ricci solitons in dimension 4. One may also prove some results as done by N. Hitchin [11] and M. Anderson [1] for complete manifolds with asymptotic flat or hyperbolic geometry. There should be a nice relationship between Ricci solitons and Yamabe constants, see the work of M. Gursky and C. LeBrun [14] for the Einstein metric cases. Since the study of Ricci solitons is a rapid growing subject, we have no intention to review the subject here. This paper is just the adaption of the previous IHES preprint (2006) [19].

#### 2. Preliminary

We first recall some basic properties about Ricci solitons (see [15,16,18,21] and [8]).

Given a compact Riemannian manifold  $(M^n, g)$  of dimension  $n \ge 3$ , we say that (M, g) is a gradient shrinking Ricci soliton if there is a smooth function *f* and a constant  $\rho > 0$  such that:

$$Rc = \rho g + D^2 f, \tag{3}$$

where *Rc* is the Ricci curvature of the metric *g*, and  $D^2 f$  is the Hessian of the potential function *f* on (*M*, *g*). We normalize f such that  $\int_M f \, dv = 0$ , where dv is the volume element of the metric g.

Taking the trace of both sides of (3), we have:

$$s = n\rho + \Delta f. \tag{4}$$

Then we have  $\int_M s = n\rho \operatorname{vol}(M) > 0$  and

$$\int_{M} s^{2} = n^{2} \rho^{2} \operatorname{vol}(M) + \int_{M} (\Delta f)^{2}.$$

Hence, the Yamabe constant in the conformal class of the metric g is positive, and the scalar curvature s is positive somewhere. Using the maximum principle, one has (see Proposition 1 in [12]) that the scalar curvature is positive everywhere in M. Actually, the scalar curvature of (M, g) is always positive according to the recent result of B.L. Chen.

Take a point  $x \in M$ . In local normal coordinates  $(x^i)$  of the Riemannian manifold (M, g) at a point x, we write the metric g as  $(g_{ij})$ . The corresponding Riemannian curvature tensor and Ricci tensor are denoted by  $Rm = (R_{ijkl})$  and  $Rc = (R_{ij})$ respectively [3]. Hence,  $R_{ij} = g^{kl}R_{ikjl}$  and  $s = g^{ij}R_{ij}$ . We write the covariant derivative of a smooth function f by  $Df = (f_i)$ , and denote the Hessian matrix of the function f by  $D^2f = (f_{ij})$ , where D is the covariant derivative of g on M. The higher-order covariant derivatives are denoted by  $f_{ijk}$ , etc. Similarly, we use the  $T_{ij,k}$  to denote the covariant derivative of the tensor  $(T_{ij})$ . We write  $T_i^i = g^{ik}T_{jk}$ . Then the Ricci soliton equation is:

$$R_{ij} = f_{ij} + \rho g_{ij}.$$

Taking covariant derivative, we get:

 $f_{ijk} = R_{ij,k}.$ 

So we have:

$$f_{ijk} - f_{ikj} = R_{ij,k} - R_{ik,j}.$$

By the Ricci formula that:

$$f_{ijk} - f_{ikj} = R^l_{ijk} f_l,$$

we obtain that:

$$R_{ij,k} - R_{ik,j} = R_{iik}^l f_l.$$

Recall that the contracted Bianchi identity is:

$$R_{ij,j}=\frac{1}{2}s_i.$$

Upon taking the trace of the previous equation, we get that:

 $\frac{1}{2}s_i + R_i^k f_k = 0,$ 

i.e.,

$$s_k = -2R_k^j f_j.$$

Then at x,

$$D_k (|Df|^2 + R + 2\rho f) = 2f_j (f_{jk} - R_{jk} + 2cg_{jk}) = 0$$

So,

$$|Df|^2 + s + 2\rho f = \Lambda, \tag{6}$$

(5)

where  $\Lambda$  is a constant. Then we have:

$$\Lambda \operatorname{vol}(M) = \int_{M} |Df|^{2} + \int_{M} s = \int_{M} |Df|^{2} + n\rho \operatorname{vol}(M).$$

This gives us that:

$$\int_{M} |Df|^{2} = (\Lambda - n\rho) \operatorname{vol}(M).$$
(7)

This implies that  $\Lambda \ge n\rho$ .

## 3. Basic facts on 4-dimensional geometry

In this section, we assume that n = 4. The basic references for the four-dimensional geometry are the papers of Atiyah, Hitchin, and Singer [2], and the books of A. Besse [4], Freed and Uhlenbeck [13], and Donaldson and Kronheimer [10]. Let W be the Weyl tensor, which is a conformal invariant, and let A be the Weyl–Schouten tensor:

$$A=Rc-\frac{s}{2(n-1)}g.$$

Then

$$Rm=W\oplus\frac{1}{n-2}A\odot g.$$

Here  $\odot$  is the Kulkarni–Nomizu product (see [4, 1.110]). Define  $\sigma_2(A) := \sigma_2(g)$  the 2nd symmetric function of the eigenvalues of the matrix *A*. Then we have:

$$\sigma_2(A) = \frac{1}{2} \left( \left| \operatorname{tr}(A) \right|^2 - |A|^2 \right) = \frac{1}{6} s^2 - \frac{1}{2} |Rc|^2.$$

Assume from now on that n = 4. Then by using the Chern–Gauss–Bonnet formula [4], the Euler number of M is given by the formula:

$$8\pi^2\chi(M) = \int_M \left(\sigma_2(A) + |W|^2\right) \mathrm{d}\nu.$$

By this formula, we see that the integral  $\int_M \sigma_2(A)$  is a conformal invariant.

Using the Hodge star operator we can write by W into self-dual and anti-self-dual part,  $W = W_+ + W_-$ . Then we have the following Hirzebruch formula [4] for the signature of the manifold M,

$$12\pi^{2}\tau(M) = \int_{M} \left( |W_{+}|^{2} - |W_{-}|^{2} \right) \mathrm{d}\nu.$$

Therefore, we have:

$$2\chi(M) \pm 3\tau(M) = \frac{1}{4\pi^2} \int_M \left( \sigma_2(A) + 2|W_{\pm}|^2 \right) d\nu_g.$$
(8)

We make two remarks below. The first remark is that people often write the Chern-Gauss-Bonnet formula in dimension 4 (see [4]) as

$$8\pi^{2}\chi(M) = \int_{M} \left(\frac{s^{2}}{24} + |W|^{2} - \frac{|B|^{2}}{2}\right) d\nu,$$

where B is the trace-free part of the Ricci curvature Rc. For our gradient shrinking Ricci soliton, we then have, by an elementary computation, that:

$$\int_{M} |B|^2 = \frac{1}{4} \int_{M} |\Delta f|^2.$$

The other remark is that the assumption (2) is equivalent to:

$$\int_{M} \left(\Delta f\right)^2 \leqslant 8\rho^2 \operatorname{vol}(M).$$

#### 4. Proofs of Theorem 1 and its corollary

Assume that (M, g) is a gradient shrinking Ricci soliton of dimension four. Then we have:

$$\int_{M} |Rc|^{2} = \int_{M} |D^{2}f + \rho g|^{2} = \int_{M} (|D^{2}f|^{2} + 2\rho\Delta f) + 4\rho^{2} \operatorname{vol}(M) = \int_{M} |D^{2}f|^{2} + 4\rho^{2} \operatorname{vol}(M)$$

Using integration by part, (6), (7), and the Ricci formula, we have:

$$\begin{split} \int_{M} \left| D^{2} f \right|^{2} &= -\int_{M} f_{iji} f_{j} = -\int_{M} \left( (\Delta f)_{j} f_{j} + R_{ij} f_{i} f_{j} \right) = \int_{M} (\Delta f)^{2} - \int_{M} \left( f_{ij} f_{i} f_{j} + \rho |Df|^{2} \right) \\ &= \int_{M} (\Delta f)^{2} + \frac{1}{2} \int_{M} |Df|^{2} \Delta f - \int_{M} \rho |Df|^{2} = \int_{M} (\Delta f)^{2} + \frac{1}{2} \int_{M} (\Lambda - s - 2\rho f) \Delta f - \int_{M} \rho |Df|^{2} \\ &= \int_{M} (\Delta f)^{2} + \frac{1}{2} \int_{M} (4\rho - s) \Delta f = \frac{1}{2} \int_{M} (\Delta f)^{2} = \frac{1}{2} \int_{M} (s - 4\rho)^{2} = \frac{1}{2} \int_{M} s^{2} - 8\rho^{2} \operatorname{vol}(M). \end{split}$$

This also implies that:

$$\int_{M} s^2 \ge 16\rho^2 \operatorname{vol}(M).$$

Then,

$$\int_{M} |Rc|^{2} = \frac{1}{2} \int_{M} s^{2} - 4\rho^{2} \operatorname{vol}(M).$$

Hence, by the relation  $\sigma_2(A) = \frac{1}{6}s^2 - \frac{1}{2}|Rc|^2$ , we have:

$$\int_{M} \sigma_2(A) = -\frac{1}{12} \int_{M} s^2 + 2\rho^2 \operatorname{vol}(M).$$
(9)

Combining (8) and (9) we get (1).

Assume that  $(M^4, g)$  is a natural oriented Kähler surface. Then it is well known (see Proposition 16.62 in [4] or see [9]) that:

$$|W_+|^2 = \frac{1}{24}s^2.$$

Hence, we have:

$$2\chi(M) + 3\tau(M) = \frac{1}{4\pi^2} \int_M \left( \sigma_2(A) + \frac{1}{12} s^2 \right).$$

It is clear from the formulae above that if (M, g) is a natural oriented Kähler surface, then we have by (9):

$$2\chi(M) + 3\tau(M) = \frac{\rho^2 \operatorname{vol}(M)}{2\pi^2}.$$

This completes the proof of the Main Theorem.

**Proof of Corollary 2.** Recall our assumption (2) that  $\int_M s^2 \leq 24\rho^2 \operatorname{vol}(M)$ . From (9), we have the following inequality:

$$\int_{M} \sigma_2(A) \ge 0.$$

Then by (8), we have:

$$2\chi(M) \pm 3\tau(M) \ge \frac{1}{2\pi^2} \int_M |W_{\pm}|^2 \ge 0,$$

which is the desired result.  $\Box$ 

We remark that it is interesting to know which closed manifolds admit Riemannian metrics with inequality  $\int_M \sigma_2(A) d\nu \ge 0$ . For this topic, one may see the work of A. Chang, M. Gursky, and P. Yang [7]. As a corollary of Chang-Gursky-Yang's result and the argument of Theorem 1, we conclude:

**Proposition 3.** Given a compact shrinking Ricci soliton (M, g) of dimension four with condition (2). There is a conformal metric  $\bar{g} = e^{2w}g$  such that  $\sigma_2(\bar{g}) > 0$ .

#### Acknowledgements

Theorem 1 was obtained in the fall of 2005 and it was reported by the author at the International Conference on Global Differential Geometry of the satellite meeting of ICM2006 held in Muenster, Germany. Prof. Huaidong Cao showed his interest to this paper, and we had tried to remove the condition (2) in Theorem 1 during his visit to Tsinghua University, Beijing in 2005. The author would like to thank Prof. S. Donaldson, Prof. Huaidong Cao, Prof. Xianzhe Dai, and Prof. C. LeBrun for their interest. The author thanks the referee very much for helpful suggestions, and he would like to thank IHES, France, for hospitality during his visit in 2006.

#### References

- [1] Michael T. Anderson, Einstein metrics with prescribed conformal infinity on 4-manifolds, arXiv:math.DG/0105243.
- [2] M. Atiyah, N. Hitchin, I. Singer, Self-duality in four-dimensional Riemannian geometry, Proc. R. Soc., Math. Phys. Eng. Sci. 362 (1978) 425–461.
- [3] T. Aubin, Non-linear Analysis on Manifolds, Springer, New York, 1982.
- [4] A. Besse, Einstein Manifolds, Springer, Berlin, 1987.
- [5] R. Bryant, Gradient Kähler-Ricci solitons, arXiv:math.DG/0407453, 2004.
- [6] H.D. Cao, Existence of gradient Kähler–Ricci soliton, in: B. Chow, R. Gulliver, S. Levy, J. Sullivan, A.K. Peters (Eds.), Elliptic and Parabolic Methods in Geometry, 1996, pp. 1–6.
- [7] A. Chang, M. Gursky, P. Yang, An equation of Monge–Ampère type in conformal geometry, and four-manifolds of positive Ricci curvature, Ann. Math. 155 (2002) 709–787.
- [8] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, L. Ni, The Ricci Flow: Techniques and Applications. Part I. Geometric Aspects, Mathematical Surveys and Monographs, vol. 135, American Mathematical Society, Providence, RI, 2007.

- [9] A. Derzinski, Self-dual Kähler manifolds and Einstein manifolds of dimension four, Compos. Math. 49 (1983) 405-433.
- [10] S.K. Donaldson, P.B. Kronheimer, The Geometry of 4-Manifold, Clarendon, Oxford, UK, 1990.
- [11] N. Hitchin, Einstein metrics and the eta-invariant, Boll. Unione Mat. Ital. IIB (Suppl. 2) (1997) 95–105.
- [12] T. Ivey, Ricci solitons on compact three manifolds, Differ. Geom. Appl. 3 (1993) 301-307.
- [13] D. Freed, K. Uhlenbeck, Instantons and 4-Manifolds, MSRI Publications, vol. 1, Springer-Verlag, 1984.
- [14] M.J. Gursky, C. LeBrun, Yamabe invariants and Spin<sup>c</sup> structures, Geom. Funct. Anal. 8 (1998) 965–977.
- [15] R. Hamilton, The formation of singularities in the Ricci flow, Surv. Differ. Geom. 2 (1995) 7–136.
- [16] B. Kleiner, J. Lott, Notes on Perelman's papers, http://www.math.lsa.umich.edu/research/ricciflow/perelman.html.
- [17] Claude LeBrun, Einstein metrics, four-manifolds, and differential topology, in: S.-T. Yau (Ed.), Papers in Honor of Calabi, Lawson, Siu, and Uhlenbeck, in: Surveys in Differential Geometry, vol. VIII, International Press of Boston, 2003, pp. 235–255.
- [18] Li Ma, Some properties of non-compact complete Riemannian manifolds, Bull. Sci. Math. 130 (2006) 330-336.
- [19] Li Ma, Remarks on compact shrinking Ricci solitons of dimension four, 2006, IHES preprint.
- [20] Li Ma, Anqiang Zhu, Nonsingular Ricci flow on a noncompact manifold in dimension three, C. R. Acad. Sci. Paris, Ser. I 137 (2009) 185-190.
- [21] L. Ni, Ancient solutions to Kähler-Ricci flow, Math. Res. Lett. 11 (2005) 10001-10020.
- [22] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv:math.DG/0211159.
- [23] X.J. Wang, X.H. Zhu, Kähler-Ricci solitons on toric manifolds with positive first Chern class, Adv. Math. 188 (2004) 87-103.
- [24] S.T. Yau, On Calabi's conjecture and some new results in algebraic geometry, Proc. Natl. Acad. Sci. USA 74 (1977) 1798–1799.