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C. R. Acad. Sci. Paris, Ser. I



Complex analysis/Analytic geometry

Algebraic approximation of analytic subsets of $\mathbb{C}^q \times \{0\}$ in \mathbb{C}^{q+1}



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| ARTICLE INFO | ABSTRACT |
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| Article history: Received 27 September 2013 Accepted after revision 16 October 2013 Available online 30 October 2013 Presented by Jean-Pierre Demailly | We prove that every analytic subset of $\mathbb{C}^q \times \{0\}$ admits approximation by algebraic sets in \mathbb{C}^{q+1} . © 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved. R É S U M É |
| | On démontre que tout sous-ensemble analytique de $\mathbb{C}^q \times \{0\}$ possède une approximation par un sous-ensemble algébrique de \mathbb{C}^{q+1} . © 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved. |

1. Introduction

The problem of algebraic approximation of holomorphic maps and its generalizations have been studied by several mathematicians (see [7–10,12–14,17,19], and references therein).

A basic question in complex analytic geometry is whether complex analytic sets can be approximated by algebraic ones. For analytic sets with isolated singularities, algebraic approximation is closely related to the problem of transforming biholomorphically a given set onto an algebraic one (see [3,7,13,16,18]). In general, there exist germs of complex analytic sets that are not biholomorphically equivalent to any germ of an algebraic set (see [23]), and only topological equivalence between analytic and algebraic set germs holds true (cf. [15]). Nevertheless, every analytic subset of $D \times \mathbb{C}^p$ of pure dimension *n* with proper projection onto *D*, where $D \subset \mathbb{C}^n$ is a Runge domain, can be approximated (in the sense of chains; for definition see Section 2) by algebraic sets of pure dimension *n* (cf. [5]). Let us recall that a domain of holomorphy $\Omega \subset \mathbb{C}^q$ is called a Runge domain if every function holomorphic on Ω can be uniformly approximated on compact subsets of Ω by polynomials in *q* complex variables (cf. [11], pp. 36, 52).

The aim of the present note is to prove the following:

Theorem 1.1. Let $\Omega \subset \mathbb{C}^q$ be a Runge domain and let X be an analytic subset of $\Omega \times \{0\} \subset \mathbb{C}^q \times \mathbb{C}$ of pure dimension n. Then there is a sequence (X_v) of algebraic subsets of $\mathbb{C}^q \times \mathbb{C}$ of pure dimension n such that $(X_v \cap (\Omega \times \mathbb{C}))$ converges to X in the sense of chains.

Observe that in Theorem 1.1, $X \subset \Omega \times \mathbb{C}$ has proper projection onto Ω , hence the theorem follows immediately from Theorem 3.1 proved in Section 3.

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¹ The research was partially supported by the NCN grant 2011/01/B/ST1/03875.

¹⁶³¹⁻⁰⁷³X/\$ – see front matter © 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved. http://dx.doi.org/10.1016/j.crma.2013.10.011

The problem of global algebraic approximation has been considered also in real geometry. In [2] it is proved that every compact smooth submanifold of \mathbb{R}^q is ε -isotopic to a nonsingular algebraic variety in \mathbb{R}^{q+1} .

2. Preliminaries

2.1. Convergence of closed sets and holomorphic chains

Let *U* be an open set in \mathbb{C}^q . By a *holomorphic chain* in *U* we mean a formal sum $A = \sum_{j \in J} \alpha_j C_j$, where α_j are nonzero integers and $\{C_j\}_{j \in J}$ is a locally finite family of pairwise distinct irreducible analytic subsets of *U* (see [6,21]; cf. [4]). The set $\bigcup_{j \in J} C_j$ is called the *support* of *A* and is denoted by |A|, whereas the C_j 's are called the *components* of *A* with *multiplicities* α_j . The chain *A* is called *positive* if $\alpha_j > 0$ for all $j \in J$. If all the components of *A* have the same dimension *n*, then *A* is called an *n*-*chain*.

Below we introduce convergence in the sense of holomorphic chains in *U*. To do this, we will need first the notion of the local uniform convergence of closed sets: let *X* and $\{X_{\nu}\}_{\nu \in \mathbb{N}}$ be closed subsets of *U*. We say that the sequence (X_{ν}) converges to *X* locally uniformly when the following two conditions hold:

- (11) for every $a \in X$ there exists a sequence (a_{ν}) such that $a_{\nu} \in X_{\nu}$ and (a_{ν}) converges to a in the Euclidean topology on \mathbb{C}^{q} ;
- (12) for every compact subset *K* of *U* such that $K \cap X = \emptyset$, one has $K \cap X_{\nu} = \emptyset$ for almost all ν .

Then we write $X_{\nu} \rightarrow X$. For the topology of local uniform convergence, see [22].

Let Z and $\{Z_{\nu}\}_{\nu \in \mathbb{N}}$ be positive *n*-chains in U. We say that the sequence (Z_{ν}) converges to Z, when:

- (c1) $|Z_{\nu}| \rightarrow |Z|$, and
- (c2) for every regular point *a* of |Z| and every submanifold *T* of *U* of dimension q n transversal to |Z| at *a* such that \overline{T} is compact and $|Z| \cap \overline{T} = \{a\}$, one has deg $(Z_{\nu} \cdot T) = \deg(Z \cdot T)$ for almost all ν .

Then we write $Z_{\nu} \rightarrow Z$. (By $Z \cdot T$ we denote the intersection product of Z and T (see, e.g., [21]).) The following lemma from [21] will be useful to us.

Lemma 2.1. Let $n \in \mathbb{N}$, and let Z and $\{Z_{\nu}\}_{\nu \in \mathbb{N}}$ be positive n-chains in U. If $|Z_{\nu}| \to |Z|$, then the following conditions are equivalent:

- (i) $Z_{\nu} \rightarrow Z$;
- (ii) for every point a from a given dense subset of the regular locus Reg(|Z|), there exists a submanifold T of U of dimension q n transversal to |Z| at a and such that \overline{T} is compact, $|Z| \cap \overline{T} = \{a\}$ and $\deg(Z_v \cdot T) = \deg(Z \cdot T)$ for almost all v.

Let now X and $\{X_{\nu}\}_{\nu \in \mathbb{N}}$ be analytic sets of pure dimension *n* in an open *U* in \mathbb{C}^{q} . We say that the sequence (X_{ν}) *converges to X in the sense of (holomorphic) chains* when the sequence (Z_{ν}) of *n*-chains converges to the *n*-chain *Z*, where *Z* and $\{Z_{\nu}\}_{\nu \in \mathbb{N}}$ are obtained by assigning multiplicity 1 to all the irreducible components of *X* and $\{X_{\nu}\}_{\nu \in \mathbb{N}}$, respectively.

2.2. Nash sets and approximation

Let Ω be an open subset of \mathbb{C}^q and let f be a holomorphic function on Ω . We say that f is a *Nash function* at $\zeta \in \Omega$ if there exist an open neighborhood U of ζ in Ω and a polynomial $P \in \mathbb{C}[Z_1, \ldots, Z_q, W]$, $P \neq 0$, such that P(z, f(z)) = 0 for $z \in U$. A holomorphic function on Ω is a Nash function if it is a Nash function at every point of Ω . A holomorphic mapping into \mathbb{C}^N is a *Nash mapping* if each of its components is a Nash function. (For more details, see [20].)

A subset *X* of Ω is called a *Nash subset* of Ω if for every $\zeta \in \Omega$ there exist an open neighborhood *U* of ζ in Ω and Nash functions f_1, \ldots, f_s on *U*, such that $X \cap U = \{z \in U : f_1(z) = \cdots = f_s(z) = 0\}$.

Our proof relies on the following facts. Let now $\Omega \subset \mathbb{C}^q$ be a Runge domain.

Theorem 2.2. (Cf. [1, Thm. 1.1].) Let X be a complex analytic subset of Ω of pure dimension n. Then, for every open Ω_0 relatively compact in Ω , there exists a sequence $(X_{\nu})_{\nu \in \mathbb{N}}$ of Nash subsets of Ω_0 of pure dimension n converging to $X \cap \Omega_0$ in the sense of holomorphic chains.

Proposition 2.1. (*Cf.* [5, *Prop. 3.2*].) Let Y be a Nash subset of $\Omega \times \mathbb{C}$ of pure dimension $k \leq q$, with proper projection onto Ω . Then there is a sequence (Y_{ν}) of algebraic subsets of $\mathbb{C}^q \times \mathbb{C}$ of pure dimension k such that $(Y_{\nu} \cap (\Omega \times \mathbb{C}))$ converges to Y in the sense of holomorphic chains.

3. Approximation by algebraic sets

The main result of this section is the following generalization of Theorem 3.1 of [5].

Theorem 3.1. Let Ω be a Runge domain in \mathbb{C}^q and let X be an analytic subset of $\Omega \times \mathbb{C}$ of pure dimension $n \leq q$ with proper projection onto Ω . Then there is a sequence (X_{ν}) of algebraic subsets of $\mathbb{C}^q \times \mathbb{C}$ of pure dimension n such that $(X_{\nu} \cap (\Omega \times \mathbb{C}))$ converges to X in the sense of chains.

Proof. Let $\{\Omega_{\nu}\}_{\nu=1}^{\infty}$ be a family of Runge domains in \mathbb{C}^q such that $\Omega_{\nu} \in \Omega_{\nu+1}$ and $\bigcup_{\nu=1}^{\infty} \Omega_{\nu} = \Omega$. Next, consider $\{\mathcal{E}_{\nu}\}_{\nu=1}^{\infty}$, where for every ν , \mathcal{E}_{ν} is a finite family of relatively compact subsets of $\Omega_{\nu+1} \times \mathbb{C}$ such that the following hold:

- (a) $\mathcal{E}_{\nu} \subset \mathcal{E}_{\nu+1}$,
- (b) $(\Omega_{\nu} \times \mathbb{C}) \cap X \subset \bigcup \{E: E \in \mathcal{E}_{\nu}\},\$
- (c) for every $E \in \mathcal{E}_{\nu}$ there are an *n*-dimensional linear subspace L of $\mathbb{C}^q \times \mathbb{C}$ and open balls E' and E'' in L and L^{\perp} , respectively, such that $E = E' \oplus E''$, and $X \cap (\overline{E'} \oplus \partial E'') = \emptyset$.

Here by L^{\perp} we denote the orthogonal complement of L in $\mathbb{C}^q \times \mathbb{C}$. The existence of \mathcal{E}_{ν} , for $\nu = 1, 2, ...$, follows from the fact that, due to the Noether normalization, for every $x \in (\overline{\Omega_{\nu} \times \mathbb{C}}) \cap X$, there are an *n*-dimensional linear subspace L of $\mathbb{C}^q \times \mathbb{C}$ and open balls E' and E'' in L and L^{\perp} , respectively, such that $E = E' \oplus E''$ is a (relatively compact) neighborhood of x in $\Omega_{\nu+1} \times \mathbb{C}$ and $X \cap (\overline{E'} \oplus \partial E'') = \emptyset$. Hence $(\Omega_{\nu} \times \mathbb{C}) \cap X$ can be covered by a finite number of E's. Let us mention that the technique of covering analytic sets by relatively compact open patches as above has been introduced in a systematic way in [4].

Remark 3.2. Let $E = E' \oplus E'' \Subset \Omega \times \mathbb{C}$ and $X \cap (\overline{E'} \oplus \partial E'') = \emptyset$, where E', E'' are open balls in L, L^{\perp} respectively, where L is an *n*-dimensional linear subspace of $\mathbb{C}^q \times \mathbb{C}$. Let (X_v) be a sequence of analytic subsets of some open neighborhood U of \overline{E} (in $\Omega \times \mathbb{C}$) of pure dimension *n* converging locally uniformly to $X \cap U$. Then, by Lemma 2.1, $X_v \cap E \rightarrowtail X \cap E$ iff generic fibers in $X_v \cap E$ and in $X \cap E$ over E' have equal cardinalities for almost all v.

Fix $\nu \in \mathbb{N}$. By Theorem 2.2, there is a sequence (\tilde{X}_{μ}) of Nash subsets of $\Omega_{\nu+1} \times \mathbb{C}$ of pure dimension n such that $\tilde{X}_{\mu} \rightarrow X \cap (\Omega_{\nu+1} \times \mathbb{C})$. This can be chosen so that there is a disc $\Delta_{r_{\nu}}$ centered at 0 of radius $r_{\nu} > \nu$, with $X \cap (\Omega_{\nu+1} \times \mathbb{C}) \subset \Omega_{\nu+1} \times \Delta_{r_{\nu}}$ and $\tilde{X}_{\mu} \subset \Omega_{\nu+1} \times \Delta_{r_{\nu}}$ for every μ . Next, it is clear that $\tilde{X}_{\mu} \cap E \rightarrow X \cap E$ for every $E \in \mathcal{E}_{\nu}$. Now there is μ such that $\hat{X}_{\nu} := \tilde{X}_{\mu}$ satisfies: dist $(\hat{X}_{\nu} \cap (\Omega_{\nu} \times \mathbb{C}), X \cap (\Omega_{\nu} \times \mathbb{C})) < \frac{1}{\nu}$, and for every $E = E' \oplus E'' \in \mathcal{E}_{\nu}$, $\hat{X}_{\nu} \cap (\overline{E'} \oplus \partial E'') = \emptyset$ and generic fibers in $\hat{X}_{\nu} \cap E$ and in $X \cap E$ over E' have equal cardinalities. (Here dist($\cdot, \cdot)$) denotes the Hausdorff distance.)

By Proposition 2.1, there is a sequence (X'_{μ}) of algebraic subsets of $\mathbb{C}^q \times \mathbb{C}$ of pure dimension n such that $X'_{\mu} \cap (\Omega_{\nu+1} \times \mathbb{C}) \rightarrow \hat{X}_{\nu}$. Therefore, there is μ such that $X_{\nu} := X'_{\mu}$ satisfies: $\operatorname{dist}(X_{\nu} \cap (\Omega_{\nu} \times \Delta_{r_{\nu}}), \hat{X}_{\nu} \cap (\Omega_{\nu} \times \Delta_{r_{\nu}})) < \frac{1}{\nu}$, and for every $E = E' \oplus E'' \in \mathcal{E}_{\nu}, X_{\nu} \cap (\overline{E'} \oplus \partial E'') = \emptyset$ and generic fibers in $X_{\nu} \cap E$ and in $\hat{X}_{\nu} \cap E$ over E' have equal cardinalities.

By the two previous paragraphs, $\operatorname{dist}(X_{\nu} \cap (\Omega_{\nu} \times \Delta_{r_{\nu}}), X \cap (\Omega_{\nu} \times \Delta_{r_{\nu}})) < \frac{2}{\nu}$, for every ν , which implies that $(X_{\nu} \cap (\Omega \times \mathbb{C}))$ converges to X locally uniformly. In view of (b), it remains to check that $X_{\nu} \cap E \rightarrow X \cap E$ for every $E \in \bigcup_{\nu=1}^{\infty} \mathcal{E}_{\nu}$. Fix $E = E' \oplus E'' \in \bigcup_{\nu=1}^{\infty} \mathcal{E}_{\nu}$ and note that, by (a), $E \in \mathcal{E}_{\nu}$ for almost all ν . Now the required assertion follows immediately by Remark 3.2, the properties of $\{\mathcal{E}_{\nu}\}_{\nu=1}^{\infty}$, and the two previous paragraphs. \Box

Acknowledgements

This paper was written during a stay of the author at Laboratoire J.A. Dieudonné, Université de Nice–Sophia Antipolis, and he wishes to thank the Laboratoire J.A. Dieudonné for their hospitality.

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