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The space $H(\text{div}, \cdot)$ on a surface – Application to Donati-like compatibility conditions on a surface





L'espace $H(div, \cdot)$ sur une surface – Application à des conditions de compatibilité du type de Donati sur une surface

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ABSTRACT

In this Note, we show how the analogue of the classical space $H(\text{div}, \cdot)$ can be defined on a surface. We then establish several properties of this space, notably the existence of a basic Green's formula satisfied by its elements. These results are then used for identifying Donati-like compatibility conditions on a surface.

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RÉSUMÉ

Dans cette Note, on montre comment définir l'analogue de l'espace classique $H(\text{div}, \cdot)$ sur une surface. On établit ensuite diverses propriétés de cet espace, en particulier l'existence d'une formule de Green fondamentale satisfaite par ses éléments. Ces résultats sont ensuite utilisés pour identifier des conditions de compatibilité du type de Donati sur une surface. © 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

In the Note [5], we showed how ad hoc *Green's formulas with little regularity on a surface*, combined with *Banach closed range theorem*, could be used for identifying Donati-like compatibility conditions on a surface. Such conditions guarantee that the components of two symmetric matrix fields $(c_{\alpha\beta})$ and $(r_{\alpha\beta})$ with $c_{\alpha\beta}$ and $r_{\alpha\beta}$ in the space $L^2(\omega)$, where ω is a domain in \mathbb{R}^2 , are the covariant components of the linearized change of metric and linearized change of curvature tensors associated with a displacement vector field of a surface $\theta(\overline{\omega})$, where $\theta: \overline{\omega} \to \mathbb{R}^3$ is a smooth immersion.

However, the displacement vector fields found in this approach satisfy either a homogeneous Dirichlet boundary condition, or a homogeneous Neumann boundary condition, on the *entire* boundary. In this Note, we use a completely different approach, essentially based on the *"surface analogue"* (cf. Section 2) of the classical space $H(\text{div}; \Omega)$, where Ω is an open subset of \mathbb{R}^n (cf. Section 2.2 in Chapter 2 of Girault and Raviart [9] or Section 3.1 in Chapter 3 of Brezzi and Fortin [3]).

This approach is in a sense more general, as it produces displacement fields that may satisfy a homogeneous Dirichlet boundary condition only on a *portion* of the boundary (Theorem 3.1).

The notations and the geometrical preliminaries are the same as in the Note [5] and, for this reason, will not be repeated here.

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Note that, in what follows, it is implicitly understood that, given a domain $\omega \subset \mathbb{R}^2$ and an immersion $\theta \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$, functions such as (for instance) $b^{\alpha}_{\sigma}, t^{\alpha\beta}|_{\beta}, \Gamma^{\sigma}_{\alpha\beta}$, etc., denote the mixed components of the second fundamental form, the covariant derivatives of a doubly contravariant tensor, the Christoffel symbols, etc., associated with the immersion θ .

Complete proofs, and applications to intrinsic shell theory, will be found in [4].

2. The "surface analogue" of the space $H(\text{div}; \cdot)$

Let there be given a domain ω in \mathbb{R}^2 and an immersion $\theta \in \mathcal{C}^2(\omega; \mathbb{R}^3)$. The "surface analogue" of the classical space $H(\text{div}; \cdot)$ is defined as:

$$\mathbb{H}(\boldsymbol{d};\omega) := \left\{ (\boldsymbol{n},\boldsymbol{m}) \in \mathbb{L}^{2}(\omega) \times \mathbb{L}^{2}(\omega); \, \boldsymbol{d}(\boldsymbol{n},\boldsymbol{m}) \in \boldsymbol{L}^{2}(\omega) \right\},\tag{1}$$

where $d(\mathbf{n}, \mathbf{m}) := (d^i(\mathbf{n}, \mathbf{m}))$ and the distributions $d^i(\mathbf{n}, \mathbf{m})$ are defined as:

$$d^{\alpha}(\boldsymbol{n},\boldsymbol{m}) := \left(n^{\alpha\beta} + b^{\alpha}_{\sigma}m^{\sigma\beta}\right)\Big|_{\beta} + b^{\alpha}_{\sigma}\left(m^{\beta\sigma}\right|_{\beta}, \qquad d^{3}(\boldsymbol{n},\boldsymbol{m}) := -m^{\alpha\beta}|_{\alpha\beta} + b^{\sigma}_{\alpha}b_{\sigma\beta}m^{\alpha\beta} + b_{\alpha\beta}n^{\alpha\beta}.$$
(2)

Naturally, the relation " $d(n, m) \in L^2(\omega)$ " appearing in the definition of the space $\mathbb{H}(d; \omega)$ is to be interpreted *in the sense* of distributions: It means that, for each $(n, m) \in \mathbb{H}(d; \omega)$, there exist vector fields $d^i(n, m) \in L^2(\omega)$ such that

$$\int_{\omega} \left(\gamma_{\alpha\beta}(\boldsymbol{\varphi}) n^{\alpha\beta} + \rho_{\alpha\beta}(\boldsymbol{\varphi}) m^{\alpha\beta} \right) \sqrt{a} \, \mathrm{d}y + \int_{\omega} \varphi_i d^i(\boldsymbol{n}, \boldsymbol{m}) \sqrt{a} \, \mathrm{d}y = 0 \quad \text{for all } \boldsymbol{\varphi} = (\varphi_i) \in \boldsymbol{\mathcal{D}}(\omega).$$

where, for any smooth enough field $\eta = (\eta_i)$,

$$\begin{split} \gamma_{\alpha\beta}(\boldsymbol{\eta}) &:= \frac{1}{2} (\eta_{\alpha|\beta} + \eta_{\beta|\alpha}) - b_{\alpha\beta}\eta_3, \\ \rho_{\alpha\beta}(\boldsymbol{\eta}) &:= \eta_{3|\alpha\beta} - b_{\alpha}^{\sigma} b_{\sigma\beta}\eta_3 + (b_{\alpha}^{\sigma})\eta_{\sigma|\beta} + (b_{\beta}^{\tau})\eta_{\tau|\alpha} + (b_{\beta|\alpha}^{\tau})\eta_{\tau} \end{split}$$

denote the covariant components of the *linearized change of metric*, and *linearized change of curvature, tensors* associated with a displacement field $\eta_i \mathbf{a}^i$ of the surface $\theta(\overline{\omega})$.

The space $\mathbb{H}(\boldsymbol{d}; \omega)$ is naturally equipped with the norm defined for each $(\boldsymbol{n}, \boldsymbol{m}) \in \mathbb{H}(\boldsymbol{d}; \omega)$ by:

$$\|(\boldsymbol{n},\boldsymbol{m})\|_{\mathbb{H}(\boldsymbol{d};\omega)} := \left(\|(\boldsymbol{n},\boldsymbol{m})\|_{\mathbb{L}^{2}(\omega)\times\mathbb{L}^{2}(\omega)}^{2} + \|\boldsymbol{d}(\boldsymbol{n},\boldsymbol{m})\|_{\boldsymbol{L}^{2}(\omega)}^{2}\right)^{1/2}$$

which clearly makes it a *Hilbert space*. We now extend the properties established in Theorems 2.4 and 2.5 of Chapter 1 of Girault and Raviart [9] for the "classical" space $H(\text{div}; \cdot)$ to the space $\mathbb{H}(\mathbf{d}; \omega)$. Note that, if the *principles* of the proofs are analogues to those of [9], substantial additional technical difficulties arise that are due to the "geometry" of the surface.

To begin with, we show that smooth fields are dense in this space.

Theorem 2.1. Let there be given a domain ω in \mathbb{R}^2 and an immersion $\theta \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$. Then the space $\mathbb{C}^{\infty}(\overline{\omega}) \times \mathbb{C}^{\infty}(\overline{\omega})$ is dense in the space $(\mathbb{H}(d; \omega), \|\cdot\|_{\mathbb{H}(d, \omega)})$.

Sketch of proof. It suffices to show that any functional $\ell \in (\mathbb{H}(\boldsymbol{d}; \omega), \|\cdot\|_{\mathbb{H}(\boldsymbol{d}; \omega)})'$ that vanishes on $\mathbb{C}^{\infty}(\overline{\omega}) \times \mathbb{C}^{\infty}(\overline{\omega})$ also vanishes on $\mathbb{H}(\boldsymbol{d}; \omega)$. By the *Riesz representation theorem*, given any functional $\ell \in (\mathbb{H}(\boldsymbol{d}; \omega), \|\cdot\|_{\mathbb{H}(\boldsymbol{d}; \omega)})'$, there exist $(\boldsymbol{s}, \boldsymbol{r}) = ((s_{\alpha\beta}), (r_{\alpha\beta})) \in \mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega)$ and $\boldsymbol{\xi} = (\xi_i) = \boldsymbol{d}(\boldsymbol{s}, \boldsymbol{r}) \in \boldsymbol{L}^2(\omega)$ such that:

$$\ell(\boldsymbol{n},\boldsymbol{m}) = \int_{\omega} \left(s_{\alpha\beta} n^{\alpha\beta} + r_{\alpha\beta} m^{\alpha\beta} \right) \sqrt{a} \, \mathrm{d}y + \int_{\omega} \xi_i d^i(\boldsymbol{n},\boldsymbol{m}) \sqrt{a} \, \mathrm{d}y$$

for all $(\boldsymbol{n}, \boldsymbol{m}) = ((n^{\alpha\beta}), (m^{\alpha\beta})) \in \mathbb{H}(\boldsymbol{d}, \omega)$. Then one first shows that the assumption that the functional ℓ vanishes on $\mathbb{C}^{\infty}(\overline{\omega}) \times \mathbb{C}^{\infty}(\overline{\omega})$ implies that:

$$(\mathbf{s}, \mathbf{r}) = (\boldsymbol{\gamma}(\boldsymbol{\xi}), \boldsymbol{\rho}(\boldsymbol{\xi})) \text{ and } \boldsymbol{\xi} = ((\xi_{\alpha}), \xi_{3}) \in \boldsymbol{H}_{0}^{1}(\omega) \times H_{0}^{2}(\omega)$$

To this end, one combines the fundamental Green's formula of Theorem 2.1 in the Note [5], an extension theorem of [6], and the *infinitesimal rigid displacement lemma of a surface* of Bernadou and Ciarlet [1] (see also Bernadou, Ciarlet and Miara [2]).

Given a functional $\ell \in (\mathbb{H}(\boldsymbol{d}; \omega), \|\cdot\|_{\mathbb{H}(\boldsymbol{d}; \omega)})'$ such that $\ell = 0$ on $\mathbb{C}^{\infty}(\overline{\omega}) \times \mathcal{C}^{\infty}(\overline{\omega})$, there thus exists a vector field $\boldsymbol{\xi} = ((\xi_{\alpha}), \xi_{3}) \in \boldsymbol{H}_{0}^{1}(\omega) \times H_{0}^{2}(\omega)$ such that:

$$\ell(\boldsymbol{n},\boldsymbol{m}) = \int_{\omega} \left(\gamma_{\alpha\beta}(\boldsymbol{\xi}) n^{\alpha\beta} + \rho_{\alpha\beta}(\boldsymbol{\xi}) m^{\alpha\beta} \right) \sqrt{a} \, \mathrm{d}y + \int_{\omega} \xi_i d^i(\boldsymbol{n},\boldsymbol{m}) \sqrt{a} \, \mathrm{d}y$$

for all $(n, m) \in \mathbb{H}(d; \omega)$. Using a simple density argument, one then shows that $\ell(n, m) = 0$ for all $(n, m) \in \mathbb{H}(d; \omega)$.

We next show that a specific *Green's formula* holds in the space $\mathbb{H}(\boldsymbol{d}; \omega)$ defined in (1).

Theorem 2.2. Let there be given a domain ω in \mathbb{R}^2 with a boundary γ of class $\mathcal{C}^{1,1}$ and an immersion $\theta \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$. (a) Let the boundary operator **b** be defined for each $(\mathbf{n}, \mathbf{m}) \in \mathbb{C}^{\infty}(\overline{\omega}) \times \mathbb{C}^{\infty}(\overline{\omega})$ by:

$$\boldsymbol{b}(\boldsymbol{n},\boldsymbol{m}) := \left(\left(b^{\alpha}(\boldsymbol{n},\boldsymbol{m}) \right), b^{3}(\boldsymbol{n},\boldsymbol{m}), b^{\nu}(\boldsymbol{n},\boldsymbol{m}) \right),$$

where:

$$b^{\alpha}(\boldsymbol{n},\boldsymbol{m}) := \left(n^{\alpha\beta} + 2b^{\alpha}_{\sigma}m^{\alpha\beta}\right)\nu_{\beta}, \quad b^{3}(\boldsymbol{n},\boldsymbol{m}) := -m^{\alpha\beta}|_{\beta}\nu_{\alpha} - \partial_{\tau}\left(m^{\alpha\beta}\nu_{\alpha}\tau_{\beta}\right), \quad b^{\nu}(\boldsymbol{n},\boldsymbol{m}) := m^{\alpha\beta}\nu_{\alpha}\nu_{\beta}. \tag{3}$$

Then the linear operator

$$(\boldsymbol{n},\boldsymbol{m}) \in \left(\mathbb{C}^{\infty}(\overline{\omega}) \times \mathbb{C}^{\infty}(\overline{\omega}); \|\cdot\|_{\mathbb{H}(\boldsymbol{d};\omega)}\right) \to \boldsymbol{b}(\boldsymbol{n},\boldsymbol{m}) \in \boldsymbol{H}^{-1/2}(\gamma) \times H^{-3/2}(\gamma) \times H^{-1/2}(\gamma)$$

is continuous. Consequently, there exists a unique continuous linear extension of this operator (denoted for convenience by the same symbol):

$$\boldsymbol{b}: \left(\mathbb{H}(\boldsymbol{d};\omega), \|\cdot\|_{\mathbb{H}(\boldsymbol{d};\omega)}\right) \to \boldsymbol{H}^{-1/2}(\gamma) \times H^{-3/2}(\gamma) \times H^{-1/2}(\gamma).$$

(b) *The following* **Greens's formula in the space** $\mathbb{H}(\boldsymbol{d}; \omega)$ *holds:*

$$\int_{\omega} \left(n^{\alpha\beta} \gamma_{\alpha\beta}(\boldsymbol{\eta}) + m^{\alpha\beta} \rho_{\alpha\beta}(\boldsymbol{\eta}) \right) \sqrt{a} \, \mathrm{d}y + \int_{\omega} d^{i}(\boldsymbol{n}, \boldsymbol{m}) \eta_{i} \sqrt{a} \, \mathrm{d}y$$

$$= {}_{H^{-1/2}(\gamma)} \langle b^{\alpha}(\boldsymbol{n}, \boldsymbol{m}), \sqrt{a} \eta_{\alpha} \rangle_{H^{1/2}(\gamma)} + {}_{H^{-3/2}(\gamma)} \langle b^{3}(\boldsymbol{n}, \boldsymbol{m}), \sqrt{a} \eta_{3} \rangle_{H^{3/2}(\gamma)}$$

$$+ {}_{H^{-1/2}(\gamma)} \langle b^{\nu}(\boldsymbol{n}, \boldsymbol{m}), \sqrt{a} \partial_{\nu} \eta_{3} \rangle_{H^{1/2}(\gamma)} \tag{4}$$

for all $(\boldsymbol{n}, \boldsymbol{m}) = ((n^{\alpha\beta}), (m^{\alpha\beta})) \in \mathbb{H}(\boldsymbol{d}; \omega)$ and all $\boldsymbol{\eta} = (\eta_i) = ((\eta_\alpha), \eta_3) \in \boldsymbol{H}^1(\omega) \times H^2(\omega)$.

Sketch of proof. Let

$$\begin{split} \left\langle \boldsymbol{b}(\boldsymbol{n},\boldsymbol{m}),\boldsymbol{\eta}\right\rangle_{\gamma} &:= {}_{H^{-1/2}(\gamma)} \left\langle \boldsymbol{b}^{\alpha}(\boldsymbol{n},\boldsymbol{m}), \sqrt{a}\eta_{\alpha} \right\rangle_{H^{1/2}(\gamma)} \\ &+ {}_{H^{-3/2}(\gamma)} \left\langle \boldsymbol{b}^{3}(\boldsymbol{n},\boldsymbol{m}), \sqrt{a}\eta_{3} \right\rangle_{H^{3/2}(\gamma)} \\ &+ {}_{H^{-1/2}(\gamma)} \left\langle \boldsymbol{b}^{\nu}(\boldsymbol{n},\boldsymbol{m}), \sqrt{a}\partial_{\nu}\eta_{3} \right\rangle_{H^{1/2}(\gamma)}, \end{split}$$
(5)

for all $(\boldsymbol{n}, \boldsymbol{m}) \in \mathbb{H}(\boldsymbol{d}; \omega)$ and all $\boldsymbol{\eta} = ((\eta_{\alpha}), \eta_3) \in \boldsymbol{H}^1(\omega) \times H^2(\omega)$, and

$$\begin{aligned} H(\gamma) &:= H^{1/2}(\gamma) \times H^{1/2}(\gamma) \times H^{3/2}(\gamma) \times H^{1/2}(\gamma), \\ H(\gamma)' &:= H^{-1/2}(\gamma) \times H^{-1/2}(\gamma) \times H^{-3/2}(\gamma) \times H^{-1/2}(\gamma). \end{aligned}$$

Using the fundamental Green's formula of Theorem 2.1 in [5], and Theorem 1.6 in Chapter 1 of Girault and Raviart [9], one then shows that there exists a constant *C* such that, for each $(n, m) \in \mathbb{C}^{\infty}(\overline{\omega}) \times \mathbb{C}^{\infty}(\overline{\omega})$,

$$\left\|\boldsymbol{b}(\boldsymbol{n},\boldsymbol{m})\right\|_{\boldsymbol{H}(\boldsymbol{\gamma})'} = \sup_{\substack{\boldsymbol{\mu} \in \boldsymbol{H}(\boldsymbol{\gamma})\\ \boldsymbol{\mu} \neq \boldsymbol{0}}} \frac{\left|\langle \boldsymbol{b}(\boldsymbol{n},\boldsymbol{m}), \boldsymbol{\mu} \rangle_{\boldsymbol{\gamma}}\right|}{\|\boldsymbol{\mu}\|_{\boldsymbol{H}(\boldsymbol{\gamma})}} \leqslant C \left\|(\boldsymbol{n},\boldsymbol{m})\right\|_{\mathbb{H}(\boldsymbol{d};\omega)}.$$

Since the space $\mathbb{C}^{\infty}(\overline{\omega}) \times \mathbb{C}^{\infty}(\overline{\omega})$ is dense in $\mathbb{H}(\mathbf{d}; \omega)$ (Theorem 2.1), the space $\mathbf{H}(\gamma)'$ is complete, and the linear operator

$$\boldsymbol{b}: \left(\mathbb{C}^{\infty}(\overline{\omega}) \times \mathbb{C}^{\infty}(\overline{\omega}), \|\cdot\|_{\mathbb{H}(\boldsymbol{d};\omega)}\right) \to \boldsymbol{H}(\boldsymbol{\gamma})'$$

is continuous (as shown above), this operator has a unique continuous linear extension to the space $\mathbb{H}(\boldsymbol{d}, \omega)$. A density argument then shows that the announced *Green's formula in the space* $\mathbb{H}(\boldsymbol{d}; \omega)$ holds. \Box

3. Application to Donati compatibility conditions on a surface

As a corollary to Theorem 2.2, we now identify and justify *Donati compatibility conditions* that are necessary and sufficient for recovering from its linearized change of metric and change of curvature tensors a displacement field that may satisfy a homogeneous Dirichlet boundary condition only on a *portion* of the boundary.

Theorem 3.1. Let there be given a domain ω in \mathbb{R}^2 with a boundary γ of class $\mathcal{C}^{1,1}$ and an immersion $\theta \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$, let γ_0 be a $d\gamma$ -measurable subset of γ such that

$$d\gamma$$
-meas $\gamma_0 > 0$,

and let there be given two tensor fields $\mathbf{c} = (c_{\alpha\beta}) \in \mathbb{L}^2(\omega)$ and $\mathbf{r} = (r_{\alpha\beta}) \in \mathbb{L}^2(\omega)$. Then there exists a vector field:

$$\boldsymbol{\zeta} = \left((\zeta_{\alpha}), \zeta_{3} \right) \in \boldsymbol{H}_{\gamma_{0}}(\omega) := \left\{ \left((\eta_{\alpha}), \eta_{3} \right) \in \boldsymbol{H}^{1}(\omega) \times H^{2}(\omega); \ \eta_{i} = \partial_{\mu} \eta_{3} = 0 \text{ on } \gamma_{0} \right\}$$

such that:

$$\gamma_{\alpha\beta}(\boldsymbol{\zeta}) = c_{\alpha\beta}$$
 and $\rho_{\alpha\beta}(\boldsymbol{\zeta}) = r_{\alpha\beta}$ in $L^2(\omega)$

if and only if:

$$\int_{\omega} \left(n^{\alpha\beta} c_{\alpha\beta} + m^{\alpha\beta} r_{\alpha\beta} \right) \sqrt{a} \, \mathrm{d}y = 0 \quad \text{for all } (\boldsymbol{n}, \boldsymbol{m}) = \left(\left(n^{\alpha\beta} \right), \left(m^{\alpha\beta} \right) \right) \in \mathbb{Y}(\omega),$$

where:

$$\mathbb{Y}(\omega) := \left\{ (\boldsymbol{n}, \boldsymbol{m}) \in \mathbb{L}^{2}(\omega) \times \mathbb{L}^{2}(\omega); \, \boldsymbol{d}(\boldsymbol{n}, \boldsymbol{m}) = \left(d^{i}(\boldsymbol{n}, \boldsymbol{m}) \right) = \boldsymbol{0} \text{ in } \mathcal{D}'(\omega) \\ and \left\langle \boldsymbol{b}(\boldsymbol{n}, \boldsymbol{m}), \boldsymbol{\eta} \right\rangle_{\mathcal{U}} = \boldsymbol{0} \text{ for all } \boldsymbol{\eta} \in \boldsymbol{H}_{\gamma_{0}}(\omega) \right\},$$

the distributions $d^{i}(\mathbf{n}, \mathbf{m})$, the boundary operator **b**, and the duality bracket $\langle \mathbf{b}(\mathbf{n}, \mathbf{m}), \eta \rangle_{\gamma}$ being defined as in (2), (3), and (5), respectively.

If this is the case, such a vector field $\boldsymbol{\zeta} \in \boldsymbol{H}_{\gamma_0}(\omega)$ is unique.

Sketch of proof. The proof of the "only if" part is a straightforward application of the *Green's formula* (4) *in the space* $\mathbb{H}(\boldsymbol{d}, \omega)$ (Theorem 2.2).

The *idea of the proof of the "if" part is the following*: Define the subspace:

 $\mathbb{X}(\omega) := \left\{ \left(\boldsymbol{\gamma}(\boldsymbol{\eta}), \boldsymbol{\rho}(\boldsymbol{\eta}) \right) \in \mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega); \ \boldsymbol{\eta} \in \boldsymbol{H}_{\gamma_0}(\omega) \right\}$

of the space $\mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega)$, considered here and subsequently as equipped with the inner product defined by (with self-explanatory notation):

$$\langle (\boldsymbol{n}, \boldsymbol{m}), (\boldsymbol{c}, \boldsymbol{r}) \rangle = \int_{\Omega} (n^{\alpha\beta} c_{\alpha\beta} + m^{\alpha\beta} r_{\alpha\beta}) \sqrt{a} \, \mathrm{d} \boldsymbol{y}$$

The proof then consists in showing that:

 $\mathbb{X}(\omega) = \mathbb{Y}(\omega)^{\perp} := \left\{ (\boldsymbol{c}, \boldsymbol{r}) \in \mathbb{L}^{2}(\omega) \times \mathbb{L}^{2}(\omega); \left\langle (\boldsymbol{n}, \boldsymbol{m}), (\boldsymbol{c}, \boldsymbol{r}) \right\rangle = 0 \text{ for all } (\boldsymbol{n}, \boldsymbol{m}) \in \mathbb{Y} \right\},\$

since the equality $\mathbb{X}(\omega) = \mathbb{Y}(\omega)^{\perp}$ is exactly what the "if" part asserts.

To this end, one first shows that the space $\mathbb{X}(\omega)$ is a closed subspace of the space $\mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega)$, as a consequence of the Korn's inequality on a surface of [1] (see also [2]).

The equality $\mathbb{Y}(\omega) = \mathbb{X}(\omega)^{\perp}$ then follows from the Green's formula (4) in the space $\mathbb{H}(\mathbf{d}, \omega)$ (Theorem 2.2).

Since the equality $\mathbb{Y}(\omega) = \mathbb{X}(\omega)^{\perp}$ holds, and since the space $\mathbb{X}(\omega)$ is closed, the equality $\mathbb{X}(\omega) = \mathbb{Y}(\omega)^{\perp}$ also holds. This establishes the "if" part.

The announced uniqueness of the vector field $\boldsymbol{\zeta} \in \boldsymbol{H}_{\gamma_0}(\omega)$ follows from the assumption that $d\gamma$ -meas $\gamma_0 > 0$.

Note that the Donati compatibility conditions of Theorem 3.1 may be viewed as extensions to surfaces of the "threedimensional" compatibility conditions of Geymonat and Suquet [8] and Geymonat and Krasucki [7].

In the special case where $\gamma_0 = \gamma$, the space $\mathbb{Y}(\omega)$ reduces to the space

$$\{(\boldsymbol{n},\boldsymbol{m})\in\mathbb{L}^2(\omega)\times\mathbb{L}^2(\omega);\,\boldsymbol{d}(\boldsymbol{n},\boldsymbol{m})=\boldsymbol{0}\,\,\mathrm{in}\,\,\boldsymbol{\mathcal{D}}'(\omega)\},\,$$

since it is clear in this case that

$$\langle \boldsymbol{b}(\boldsymbol{n},\boldsymbol{m}),\boldsymbol{\eta} \rangle_{\mathcal{H}} = 0$$
 for all $\boldsymbol{\eta} = ((\eta_{\alpha}),\eta_{3}) \in \boldsymbol{H}_{0}^{1}(\omega)H_{0}^{2}(\omega).$

Therefore, when $\gamma_0 = \gamma$, the necessary and sufficient condition of Theorem 3.1 is *the same* as that already found in Theorem 4.1 of [5] by means of a completely different approach.

As another corollary to Theorem 2.2, one can also identify and justify *Donati compatibility conditions* that are necessary and sufficient for recovering from its linearized change of metric and change of curvature tensors a displacement field that satisfies a *homogeneous Neumann boundary condition on the entire boundary*; cf. [4]. Note that the Donati conditions found in this fashion are *different* from those found in this case in Theorem 3.2 of [5], however.

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