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A theorem on the existence of symmetries of fractional PDEs

Un théorème sur l'existence de symétries pour les équations aux derivées partielles fractionnaires

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ABSTRACT

We propose a theorem that extends the classical Lie approach to the case of fractional partial differential equations (fPDEs) of the Riemann–Liouville type in $(1+1)$ dimensions.

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RÉSUMÉ

Nous proposons un théorème qui generalise la méthode classique de Lie à l'étude d'équations aux derivées partielles fractionnaires de type Riemann–Liouville en $(1+1)$ dimensions.

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1. Introduction

The aim of this paper is to establish a general approach for the determination of Lie symmetries for fractional differential equations (FDEs) in $(1+1)$ dimensions. Since the works of Abel, Riemann, Liouville, etc. in the XIXth century [7] these equations have been largely investigated. Especially in the last decade, there has been a resurgence of interest, due to their manifold applications in statistical mechanics, economics, social sciences, and nonlinear phenomena like anomalous diffusion.

The main theorem proposed here concerning the existence of symmetries for FDEs generalizes the very few results known in the literature. In [1,2], the case of equations involving fractional derivatives with respect to one independent variable has been considered. In [3,4], interesting scale invariant solutions of diffusion equations have been constructed. The intrinsic *noncommutativity* of the fractional derivatives with respect to different variables, and—in the case of a single variable—with respect to different fractional orders, has represented until now the main problem in the treatment of symmetries of fractional PDEs.

Our strategy is inspired by the classical Lie theory: the annihilation of the prolonged action of the vector fields generating the symmetry transformations is imposed. This condition leads to a system of determining equations that allow us to deduce the explicit expression for the symmetry generators. The knowledge of the invariants associated with such generators is a sufficient condition to reduce a given fractional partial differential equation into a new one, characterized by a smaller

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number of independent variables. In the case of a fractional ODE, the reduction process leads to another fractional ODE of reduced order.

In this paper, we shall focus on the case of the Riemann–Liouville fractional calculus.

Let $AC(\Omega)$ be the space of absolutely continuous functions on the interval $\Omega := [a,b] \subset \mathbb{R}$. We denote by $AC^n(\Omega)$, $n \in \mathbb{N}$, the space of functions $f : \Omega \to \mathbb{R}$ such that $f \in C^{n-1}(\Omega)$ and $\frac{\mathrm{d}^{n-1} f}{\mathrm{d}x^{n-1}}(x) \in AC(\Omega)$.

Definition 1.1 *(Riemann–Liouville fractional operator).* Let $p \in \mathbb{R}^+$ and $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$, with $f \in AC^{[p]+1}([a,b])$, $[p] \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ such that $[p] \leqslant p < [p]+1$. Let $t \in (a,b)$. The Riemann–Liouville fractional integral of order p and terminals (a,t) is defined by:

$$ {}_a\mathcal{D}_t^{-p} f(t) := \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} f(\tau)\,\mathrm{d}\tau, \tag{1} $$

whereas the Riemann–Liouville fractional derivative of order p and terminals (a,t) is defined by:

$$ {}_a\mathcal{D}_t^{p} f(t) := \frac{\mathrm{d}^{[p]+1}}{\mathrm{d}t^{[p]+1}} {}_a\mathcal{D}_t^{p-[p]-1} f(t) = \frac{1}{\Gamma(1+[p]-p)} \frac{\mathrm{d}^{[p]+1}}{\mathrm{d}t^{[p]+1}} \int_a^t (t-\tau)^{[p]-p} f(\tau)\,\mathrm{d}\tau, \tag{2} $$

where $\Gamma(x) := \int_0^\infty t^{x-1} \mathrm{e}^{-t}\,\mathrm{d}t$ is the Gamma function.

When $p \equiv k \in \mathbb{N}$, the previous definitions coincide with the usual kth-fold integral ($\lim_{p\to k^\pm} {}_a\mathcal{D}_t^{-p} f(t) = \int_a^t \mathrm{d}\tau_1 \int_a^{\tau_1} \mathrm{d}\tau_2 \cdots \int_a^{\tau_{k-1}} \mathrm{d}\tau_k\, f(\tau_k) = \frac{1}{(k-1)!} \int_a^t (t-\tau)^{k-1} f(\tau)\,\mathrm{d}\tau$) and with the kth-order derivative respectively ($\lim_{p\to k^\pm} {}_a\mathcal{D}_t^{p} f(t) = \frac{\mathrm{d}^k f(t)}{\mathrm{d}t^k}$). We can now define the *partial Riemann–Liouville fractional derivative*. For the sake of simplicity, we will consider the case of a function f of two variables x_1 and x_2.

Definition 1.2 *(Partial fractional derivative and total fractional derivative).* Let $p \in \mathbb{R}^+$ and $f(x_1,x_2) : [a_1,b_1] \times [a_2,b_2] \to \mathbb{R}$, $[a_1,b_1] \subset \mathbb{R}$, $[a_2,b_2] \subset \mathbb{R}$, $f^{(k,0)}(x_1,x_2) := \frac{\partial^k f}{\partial x_1^k}(x_1,x_2)$ continuous and integrable $\forall k \in \mathbb{N}_0$ s.t. $k \leqslant [p]+1$ and $\forall x_2 \in [a_2,b_2]$. We define the partial Riemann–Liouville fractional derivative by:

$$ {}_{a_1}\partial_{x_1}^{p} f(x_1,x_2) = \frac{1}{\Gamma(1+[p]-p)} \frac{\partial^{[p]+1}}{\partial {x_1}^{[p]+1}} \int_{a_1}^{x_1} (x_1-t)^{[p]-p} f(t,x_2)\,\mathrm{d}t. \tag{3} $$

Let $g : [a_1,b_1] \to [a_2,b_2]$ a function such that $f(x_1,g(x_1))$ satisfies the requirements of Definition 1.1 respect to the variable x_1. We define the total fractional derivative with respect to the variable x_1 by:

$$ {}_a\mathcal{D}_{x_1}^{p} f\big(x_1,g(x_1)\big) := \frac{1}{\Gamma(1+[p]-p)} \frac{\mathrm{d}^{[p]+1}}{\mathrm{d}{x_1}^{[p]+1}} \int_a^{x_1} (x_1-\tau)^{[p]-p} f\big(\tau,g(\tau)\big)\,\mathrm{d}\tau. \tag{4} $$

A mixed fractional derivative can be directly introduced. However, note that ${}_{a_2}\partial_{x_2}^{q}\, {}_{a_1}\partial_{x_1}^{p} f(x_1,x_2) \neq {}_{a_1}\partial_{x_1}^{p}\, {}_{a_2}\partial_{x_2}^{q} f(x_1,x_2)$.

2. Lie theory for fractional partial differential equations

Let us consider the case of FPDES (*fractional partial differential equations*) with one dependent variable $u \in U \subseteq \mathbb{R}$ and two independent variables $(x_1,x_2) \in X \subseteq \mathbb{R}^2$. We suppose that the equation takes the form:

$$ \mathcal{E}\big(x_1,x_2,u,{}_a\partial_{m(1),3-m(1)}^{p_1,q_1} u,\ldots,{}_a\partial_{m(K),3-m(K)}^{p_K,q_K} u\big) = 0. \tag{5} $$

Here $\mathcal{E}$ is a polynomial involving K fractional derivatives of the form ${}_a\partial_{1,2}^{p,q} u(x_1,x_2) := {}_a\partial_{x_1}^{p}\, {}_a\partial_{x_2}^{q} u(x_1,x_2)$ or ${}_a\partial_{2,1}^{p,q} u(x_1,x_2) := {}_a\partial_{x_2}^{p}\, {}_a\partial_{x_1}^{q} u(x_1,x_2)$, where $K \in \mathbb{N}$, $m(i) : \{1,\ldots,K\} \subset \mathbb{N} \to \{1,2\}$, $a \in \mathbb{R}$, and $p,q \in [0,+\infty)$ are not both zero. In the subsequent considerations, we shall assume that all fractional derivatives appearing in $\mathcal{E}$ have *the same lower extreme a*. A *continuous symmetry group G* or *Lie symmetry* for the equation $\mathcal{E} = 0$ is a one-parameter group of continuous transformations that maps solutions $(x_1,x_2,u) \in X \times U =: \mathcal{M}$ into solutions $g \cdot (x_1,x_2,u) = (\tilde{x}_1,\tilde{x}_2,\tilde{u}) = (\Xi_g(x_1,x_2,u), \Phi_g(x_1,x_2,u)) \in \mathcal{M}$, $g \in G$, for some functions $\Xi_g : \mathcal{M} \to X$, $\Phi_g : \mathcal{M} \to U$. A generic element $g \in G$ has the form $g = \mathrm{e}^{\epsilon \mathbf{v}}$, where $\epsilon \in \mathbb{R}$ is the parameter of the group transformation and $\mathbf{v}$ is a vector field generating G. We shall restrict to vector fields of the form $\mathbf{v} = \xi^1(x_1,x_2,u)\partial_{x_1} + \xi^2(x_1,x_2,u)\partial_{x_2} + \phi(x_1,x_2,u)\partial_u$, i.e. we will study *point symmetries*. We also assume that the action of the symmetry group G, $(x_1,x_2,u) \xrightarrow{g} (\tilde{x}_1,\tilde{x}_2,\tilde{u})$, can be expressed by means of smooth functions such that $\frac{\mathrm{d}\tilde{x}_i}{\mathrm{d}\epsilon}|_{\epsilon=0} =$

$\xi^i(x_1, x_2, u)$, $i = 1, 2$, $\frac{d\tilde{u}}{d\epsilon}|_{\epsilon=0} = \phi(x_1, x_2, u)$. As in the case of standard differential equations [5], we prolong the vector field as:

$$\begin{aligned}\mathrm{pr}^{(\mathcal{E})}\mathbf{v} &= \xi^1(x_1, x_2, u)\partial_{x_1} + \xi^2(x_1, x_2, u)\partial_{x_2} + \phi(x_1, x_2, u)\partial_u \\ &\quad + \sum_{\substack{l,m \in \mathbb{N}_0 \\ (l,n)\neq(0,0)}} \phi_{1,2}^{l,n}(x_1, x_2, u, \ldots)\partial_{\partial_{1,2}^{l,n}u} + \sum_i \sum_{\substack{k,r \in \mathbb{N}_0 \\ k-p_i \notin \mathbb{N},\ r-q_i \notin \mathbb{N}}} \phi_{m(i),3-m(i)}^{(p_i-k,q_i-r)}(x_1, x_2, u, \ldots)\partial_{{}_a\partial_{m(i),3-m(i)}^{p_i-k,q_i-r}u},\end{aligned} \tag{6}$$

where $m(i) = 1, 2$, the sum $\sum_i$ runs over all the ordered couples of parameters (p_i, q_i) such that at least one of the parameters selected among p_i and q_i is a non-integer positive real number and ${}_a\partial_{m(i),3-m(i)}^{p_i,q_i}u$ does appear in $\mathcal{E}$. By definition, $\phi_{m,3-m}^{(p,q)} := \frac{d}{d\epsilon}[{}_a\partial_{\tilde{x}_m}^p {}_a\partial_{\tilde{x}_{3-m}}^q \tilde{u}(\tilde{x}_1, \tilde{x}_2)]|_{\epsilon=0}$, $m = 1, 2$.

The following theorems represent the main results of the paper.

Theorem 2.1 *(Prolongation formula). Assume that G is a local group of transformations acting on $\mathcal{M} = X \times U$. Then, for $m = 1, 2$ and $p, q \in (0, +\infty)$, we have the following explicit expressions for the coefficients of the prolonged vector field (6):*

$$\phi_m^p = {}_a\mathcal{D}_m^p\phi + {}_a\mathcal{D}_m^p(u\mathrm{D}_m\xi^m) - {}_a\mathcal{D}_m^{p+1}(\xi^m u) + \xi^m {}_a\mathcal{D}_m^{p+1}u + \xi^{3-m} {}_a\mathcal{D}_m^p\partial_{3-m}u - {}_a\mathcal{D}_m^p(\xi\partial_{3-m}u), \tag{7a}$$

$$\begin{aligned}\phi_{m,3-m}^{p,q} &= {}_a\mathcal{D}_{m,3-m}^{p,q}\left(\phi - \sum_{i=1}^2 \xi^i\partial_i u\right) + \sum_{i=1}^2 \xi^i\partial_i {}_a\partial_{m,3-m}^{p,q}u + {}_a\mathcal{D}_{m,3-m}^{p,q}\mathrm{D}_{3-m}(\xi^{3-m}u) \\ &\quad + {}_a\mathcal{D}_m^p\mathrm{D}_m(\xi^m {}_a\partial_{3-m}^q u) - {}_a\mathcal{D}_{m,3-m}^{p,q+1}(\xi^{3-m}u) - {}_a\mathcal{D}_m^{p+1}(\xi^m {}_a\partial_{3-m}^q u),\end{aligned} \tag{7b}$$

where we use the notations $\partial_i := \frac{\partial}{\partial x_i}$ and $\mathrm{D}_i := \frac{\mathrm{D}}{\mathrm{D}x_i}$ for the partial and total derivative, respectively. In particular, by taking $a = 0$ and using Osler's formula [6], we have [2]:

$$\begin{aligned}\phi_m^p &= {}_0\partial_m^p\phi + {}_0\mathcal{D}_m^p u(\partial_u\phi - p\mathrm{D}_m\xi^m) + \sum_{n=2}^{\infty}\sum_{l=2}^{n}\sum_{k=2}^{l}\sum_{r=0}^{k-1}\binom{p}{n}\binom{n}{l}\binom{k}{r}\frac{x_m^{n-p}(-u)^r}{k!\Gamma(n+1-q)}\frac{d^l}{dx_m^l}(u^{k-r})\frac{\partial^{n-l+k}\phi}{\partial x_m^{n-l}\partial u^k} \\ &\quad - u_0\partial_m^p\partial_u\phi + \sum_{n=1}^{\infty}\left\{\left[\binom{p}{n}\partial_m^n\partial_u\phi - \binom{p}{n+1}\mathrm{D}_m^{n+1}\xi^m\right]{}_0\mathcal{D}_m^{p-n}u - \binom{p}{n}\mathrm{D}_m^n\xi^{3-m}\partial_{3-m}{}_0\mathcal{D}_m^{p-n}u\right\}.\end{aligned} \tag{7c}$$

Theorem 2.2 *(Symmetries for FPDEs, case $1 + 1$). Under the hypotheses of the previous theorem, given a* FPDE *of the form (5), if the relation*

$$\mathrm{pr}^{(\mathcal{E})}\mathbf{v}(\mathcal{E})\big|_{\mathcal{E}=0} = 0 \tag{8}$$

holds, then **v** *is the generator of a Lie symmetry of Eq. (5).*

As an application of the previous theory, we propose a symmetry analysis of a fractional KdV–Burgers equation.

Definition 2.1 *(Fractional KdV–Burgers equation).* We shall call the equation:

$${}_0\partial_{x_2}^p u + u\,{}_0\partial_{x_1}^q u + {}_0\partial_{x_1}^r u = 0, \quad p, q, r \in \mathbb{R}^+, \tag{9}$$

the fractional Korteweg–de Vries–Burgers equation.

This form of the fKdV–Burgers equation, at the best of our knowledge, is new. In the following, we will adopt the short notation $u^{(p,0)} := {}_0\partial_{x_1}^p u$ and $u^{(0,q)} := {}_0\partial_{x_2}^q u$. We consider the case $p, q, r \in \mathbb{R}/\mathbb{Z}$, with $q < r$. The determining equation takes the form:

$$\left(\phi_2^p + u\phi_1^q + \phi u^{(q,0)} + \phi_1^r\right)\big|_{u^{(0,p)}+uu^{(q,0)}+u^{(r,0)}=0} = 0. \tag{10}$$

We no longer have a translational symmetry. We get uniquely the symmetry generator of scaling transformations:

$$\mathbf{v} = px_1\partial_1 + rx_2\partial_2 + p(q-r)u\partial_u. \tag{11}$$

If $q = r$, it can be similarly proved that the symmetry generator is $\mathbf{v} = px_1\partial_1 + rx_2\partial_2$. If $q \neq r$, we can obtain an invariant by means of the relation $\mathbf{v}\eta(x_1, x_2, u) = 0$. Consequently, we perform a symmetry reduction by looking for a solution of the form $u(x_1, x_2) = x_2^{\frac{p(q-r)}{r}} v(x_1 x_2^{-\frac{p}{r}})$. It is easily proved that:

$$ {}_0\partial^p_{x_2} v\left(x_1 x_2^{-\alpha}\right) = x_2^{-p}\left(\mathfrak{D}^{1-p,p}_{\frac{1}{\alpha}} v\right)\left(x_1 x_2^{-\alpha}\right), \quad \alpha = p/r, \tag{12} $$

where $(\mathfrak{D}^{c,a}_b f)(y) := \prod_{j=0}^{[a]}(j + c - \frac{y}{b}\frac{\mathrm{d}}{\mathrm{d}y})\mathfrak{K}^{c+a,[a]+1-a}_b f(y)$ is the *Erdély–Kober fractional differential operator of order* $a \geqslant 0$ and $(\mathfrak{K}^{c,a}_b f)(y) := \frac{1}{\Gamma(a)}\int_1^\infty (\eta - 1)^{a-1}\eta^{-(a+c)} f(y\eta^{\frac{1}{b}})\,\mathrm{d}\eta$, $a > 0$, $b, c \in \mathbb{R}$, is the *Erdélyi–Kober fractional integral operator*. Using the relation ${}_0\partial^p_x f(\lambda x) = \lambda^p {}_0\partial^p_{\lambda x} f(\lambda x)$, one gets the reduced equation in the form:

$$ z^{q-r}\mathfrak{D}^{1-p,p}_{\frac{r}{p}}\left(z^{r-q} v(z)\right) + v(z){}_0\mathcal{D}^q_z v(z) + {}_0\mathcal{D}^r_z v(z) = 0, \quad z = x_1 x_2^{-\frac{p}{r}}. \tag{13} $$

This equation can be solved numerically. Its solutions, by means of (11), will provide invariant solutions of the KdV–Burgers equation.

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