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Partial differential equations

Low- and high-energy solutions of nonlinear elliptic oscillatory problems

Solutions à basse et haute énergie pour des problèmes elliptiques non linéaires avec terme oscillatoire

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A R T I C L E   I N F O

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A B S T R A C T

In this Note, we study the existence of low- or high-energy solutions for a class of elliptic problems containing a nonlinear term that oscillates either near the origin or at infinity. We point out the competition effect between the oscillatory nonlinearity, a polynomial growth term, and the values of a real parameter. The proofs combine related variational methods.

R É S U M É

Dans cette Note, nous étudions l'existence de solutions à basse ou à haute énergie pour une classe de problèmes elliptiques contenant un terme non linéaire oscillatoire autour de l'origine ou à l'infini. Nous mettons en évidence l'effet de compétition entre la non-linéarité oscillatoire, le terme à croissance polynomiale et les valeurs d'un paramètre réel. Les preuves combinent des méthodes topologiques et variationnelles.

Version française abrégée

Soit $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ un domaine borné et régulier, $\beta \in L^\infty(\Omega)$, $\lambda \in \mathbb{R}$, $q > 0$ et $f : [0, \infty) \to \mathbb{R}$ une fonction continue qui oscille autour de l'origine ou à l'infini. Nous supposons que $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ est une fonction continue telle que, pour tout $(x, \xi) \in \Omega \times \mathbb{R}^N$,

$$A(x, \xi) \cdot \xi \geqslant \Gamma_1 |\xi|^p \quad \text{et} \quad \left|A(x, \xi)\right| \leqslant \Gamma_2 |\xi|^{p-1},$$

où $p > 1$ et $\Gamma_1, \Gamma_2 > 0$.

Dans cette Note, nous étudions le problème non linéaire :

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$$\begin{cases} -\operatorname{div} A(x, \nabla u) = \lambda\beta(x)u^q + f(u) & \text{dans } \Omega \\ u \geqslant 0 & \text{dans } \Omega \\ u = 0 & \text{sur } \partial\Omega. \end{cases} \qquad (P)$$

Le premier résultat de cette Note porte sur le cas où f a des oscillations autour de l'origine. Nous montrons d'abord que le problème (P) a une infinité de solutions « à basse énergie » si $q \geqslant p-1$ et au moins un nombre fini de solutions si $0 < q < p-1$. Plus précisément, si $q \geqslant p-1$, nous montrons l'existence d'une suite $\{u_j\} \subset W_0^{1,p}(\Omega)$ de solutions faibles du problème (P) telle que :

$$\lim_{j\to+\infty} \|u_j\|_{W_0^{1,p}(\Omega)} = \lim_{j\to+\infty} \|u_j\|_{L^\infty(\Omega)} = 0.$$

Dans le cas où f a des oscillations à l'infini, il existe une infinité de solutions $\{u_j\} \subset W_0^{1,p}(\Omega)$ si $0 < q \leqslant p-1$ et au moins un nombre fini de solutions si $q > p-1$. De plus, si $0 < q \leqslant p-1$, alors $\lim_{j\to+\infty} \|u_j\|_{L^\infty(\Omega)} = +\infty$.

1. Introduction

Competition phenomena in elliptic equations have been widely studied in the literature in different contexts. After the seminal work [1], where Ambrosetti, Brezis and Cerami studied a Laplacian equation involving a concave–convex nonlinearity, a lot of papers appeared on this subject. Also when dealing with singular terms, the interactions with different type of nonlinearities were investigated: see, for instance, Ghoussoub and Yuan [3], Pucci and Servadei [5,6] for equations involving superlinear and subcritical terms.

In this Note, we are interested in problems driven by general operators of p-Laplacian type involving oscillatory terms, in the presence of a concave or convex power. Usually, equations involving oscillatory nonlinearities give infinitely many distinct solutions, but the presence of an additional term may alter the situation.

Let $\Omega \subset \mathbb{R}^N$ ($N \geqslant 3$) be a bounded domain with smooth boundary, $q > 0$, $\lambda \in \mathbb{R}$, and let $f : [0, +\infty) \to \mathbb{R}$ be a continuous function. Suppose that $\beta \in L^\infty(\Omega)$ is a potential that is *indefinite* in sign. We also assume that $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function such that:

$$A(x,\xi)\cdot\xi \geqslant \Gamma_1|\xi|^p \quad \text{and} \quad \left|A(x,\xi)\right| \leqslant \Gamma_2|\xi|^{p-1} \quad \text{for all } (x,\xi) \in \Omega \times \mathbb{R}^N,$$

for some $p > 1$ and $0 < \Gamma_1 \leqslant \Gamma_2$. Suppose that A derives from a potential, namely $A = \nabla_\xi a$, where $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ is continuous, $a(x,0) = 0$, $a(x,\xi) = a(x,-\xi)$ for all $(x,\xi) \in \Omega \times \mathbb{R}^N$, and $a(x,\cdot)$ is strictly convex in $\mathbb{R}^N$ for all $x \in \Omega$.

We are concerned with the nonlinear Dirichlet problem:

$$\begin{cases} -\operatorname{div} A(x, \nabla u) = \lambda\beta(x)u^q + f(u) & \text{in } \Omega \\ u \geqslant 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (1)$$

2. Oscillation near the origin

Set $F(s) := \int_0^s f(t)\,dt$ and assume that:

$$\liminf_{s\to0^+} \frac{f(s)}{s^{p-1}} =: -\ell_0 \in [-\infty, 0), \qquad -\infty < \liminf_{s\to0^+} \frac{F(s)}{s^p} \leqslant \limsup_{s\to0^+} \frac{F(s)}{s^p} = +\infty. \qquad (2)$$

Examples. (i) Assume that $\alpha, \sigma, \gamma \in \mathbb{R}$ satisfy $1 < \sigma + 1 < \alpha < p$ and $\gamma > 0$. Define:

$$f(s) = \begin{cases} \alpha s^{\alpha-1}(1-\sin s^{-\sigma}) + \sigma s^{\alpha-\sigma-1}\cos s^{-\sigma} - p\gamma s^{p-1} & \text{if } s > 0 \\ 0 & \text{if } s = 0. \end{cases} \qquad (3)$$

(ii) Assume that α, σ and $\gamma \in \mathbb{R}$ are such that $1 < \alpha < p$, $\sigma > 0$, $\alpha - \sigma > 1$ and $\gamma > 0$. Define:

$$f(s) = \begin{cases} \alpha s^{\alpha-1}\cos^2 s^{-\sigma} - 2\sigma s^{\alpha-\sigma-1}\cos s^{-\sigma}\sin s^{-\sigma} - p\gamma s^{p-1} & \text{if } s > 0 \\ 0 & \text{if } s = 0. \end{cases} \qquad (4)$$

Then the functions defined by relations (3) and (4) have oscillation near the origin, in the sense described by hypothesis (2).

The main result in this section is the following.

Theorem 2.1. *Assume that f satisfies condition (2). If either*

a) $q = p-1$, $\ell_0 \in (0, +\infty)$ *and* $\lambda\beta(x) < \lambda_0$ *a.e.* $x \in \Omega$ *for some* $\lambda_0 \in (0, \ell_0)$ *or*
b) $q = p-1$, $\ell_0 = +\infty$ *and* $\lambda \in \mathbb{R}$ *is arbitrary or*
c) $q > p-1$ *and* $\lambda \in \mathbb{R}$ *is arbitrary,*

then there exists a sequence $\{u_j\}_j$ *in* $W_0^{1,p}(\Omega)$ *of distinct weak solutions of problem* (1) *such that*:

$$\lim_{j\to+\infty}\|u_j\|_{W_0^{1,p}(\Omega)}=\lim_{j\to+\infty}\|u_j\|_{L^\infty(\Omega)}=0.$$

Assume that $0<q<p-1$. *Then for every* $k\in\mathbb{N}$ *there exists* $\Lambda_k>0$ *such that problem* (1) *has at least k distinct weak solutions* $u_1,\dots,u_k\in W_0^{1,p}(\Omega)$ *such that* $\|u_j\|_{W_0^{1,p}(\Omega)}\leqslant 1/j$ *and* $\|u_j\|_{L^\infty(\Omega)}\leqslant 1/j$ *for all* $j=1,\dots,k$, *provided that* $|\lambda|<\Lambda_k$.

Sketch of the proof. Consider the auxiliary problem

$$\begin{cases} -\operatorname{div} A(x,\nabla u)+K(x)|u|^{p-2}u=h(x,u) & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Throughout this Note we assume that $K\in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_{x\in\Omega}K(x)>0$, while $h:\Omega\times[0,+\infty)\to\mathbb{R}$ is a Carathéodory function satisfying $h(x,0)=0$ for a.e. $x\in\Omega$. Set $H(x,s):=\int_0^s h(x,t)\,dt$, for all $s\in\mathbb{R}$.

A key ingredient in the proof of Theorem 2.1 if $q\geqslant p-1$ is the following multiplicity property.

Lemma 2.2. *Assume that the following hypotheses are fulfilled*:

$$\text{there exists } \bar{s}>0 \text{ such that } \sup_{s\in[0,\bar{s}]}\bigl|h(\cdot,s)\bigr|\in L^\infty(\Omega); \tag{6}$$

$$\text{there exist two sequences } \{\delta_j\}_j \text{ and } \{\eta_j\}_j \text{ with } 0<\eta_{j+1}<\delta_j<\eta_j \text{ and } \lim_{j\to+\infty}\eta_j=0 \text{ such that } h(x,s)\leqslant 0$$
$$\text{for a.e. } x\in\Omega \text{ and for every } s\in[\delta_j,\eta_j],\ j\in\mathbb{N}; \tag{7}$$

$$-\infty<\liminf_{s\to0^+}\frac{H(x,s)}{s^p}\leqslant\limsup_{s\to0^+}\frac{H(x,s)}{s^p}=+\infty \text{ uniformly for a.e. } x\in\Omega. \tag{8}$$

Then there exists a sequence $\{u_j\}_j\subset W_0^{1,p}(\Omega)$ *of distinct non-trivial non-negative weak solutions of problem* (5) *such that* $\lim_{j\to+\infty}\|u_j\|_{W_0^{1,p}(\Omega)}=\lim_{j\to+\infty}\|u_j\|_{L^\infty(\Omega)}=0$.

Returning to the proof of Theorem 2.1, let us first assume that $q=p-1$, $\ell_0\in(0,+\infty)$, and $\lambda\in\mathbb{R}$ is such that $\lambda\beta(x)<\lambda_0$ a.e. $x\in\Omega$ for some $\lambda_0\in(0,\ell_0)$. Let us choose $\tilde{\lambda}_0\in(\lambda_0,\ell_0)$ and let $K(x):=\tilde{\lambda}_0-\lambda\beta(x)$ and $h(x,s):=\tilde{\lambda}_0 s^{p-1}+f(s)$.

Next, we assume that $q=p-1$, $\ell_0=+\infty$, and $\lambda\in\mathbb{R}$. In this case we choose $\tilde{\lambda}_0\in(\lambda_0,\ell_0)$ and set $K(x):=\tilde{\lambda}_0$ and $h(x,s):=(\lambda\beta(x)+\tilde{\lambda}_0)s^{p-1}+f(s)$.

If $q>p-1$ and $\lambda\in\mathbb{R}$, we take $\tilde{\lambda}_0\in(0,\ell_0)$ and define $K(x):=\tilde{\lambda}_0$ and $h(x,s):=\lambda\beta(x)s^q+\tilde{\lambda}_0 s^{p-1}+f(s)$.

In all these cases, by straightforward computation, we deduce that K and h satisfy the assumptions of Lemma 2.2. Thus, problem (5) has infinitely many solutions $\{u_j\}_j$ satisfying $\lim_{j\to+\infty}\|u_j\|_{W_0^{1,p}(\Omega)}=\lim_{j\to+\infty}\|u_j\|_{L^\infty(\Omega)}=0$. Due to the choice of K and h, we also obtain that u_j is a weak solution of problem (1).

Let us now assume that $0<q<p-1$. We associate with problem (5) the energy functional $\mathcal{E}_{K,h}:W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by $\mathcal{E}_{K,h}(u)=\int_\Omega a(x,\nabla u(x))\,dx+\frac{1}{p}\int_\Omega K(x)|u(x)|^p\,dx-\int_\Omega H(x,u(x))\,dx$.

The key ingredient in this case is the following result.

Lemma 2.3. *Assume that the following hypotheses are fulfilled*:

$$\text{there exists } M>0 \text{ such that } \bigl|h(x,s)\bigr|\leqslant M \text{ for a.e. } x\in\Omega \text{ and for any } s\geqslant0; \tag{9}$$

$$\text{there exist } \delta \text{ and } \eta, \text{ with } 0<\delta<\eta, \text{ such that } h(x,s)\leqslant0 \text{ for a.e. } x\in\Omega \text{ and for any } s\in[\delta,\eta]. \tag{10}$$

Then

i) *the functional* $\mathcal{E}_{K,h}$ *is bounded from below on* W^η *and its infimum is attained at some* $u_\eta\in W^\eta$, *where* $W^\eta:=\{u\in W_0^{1,p}(\Omega):\ \|u\|_{L^\infty(\Omega)}\leqslant\eta\}$, *and* η *is the positive parameter given in* (10);
ii) $u_\eta\in[0,\delta]$, *where* δ *is the positive parameter given in* (10);
iii) u_η *is a non-negative weak solution of problem* (5).

Fix $\tilde{\lambda}_0\in(0,\ell_0)$ and define $K(x):=\tilde{\lambda}_0$ and $h(x,s,\lambda):=\lambda\beta(x)s^q+\tilde{\lambda}_0 s^{p-1}+f(s)$. Using the fact that $h(x,s,0)=\tilde{\lambda}_0 s^{p-1}+f(s)$, we deduce that there exist sequences $\{\delta_j\}_j$, $\{\eta_j\}_j$, $\{s_j\}_j$ and $\{\lambda_j\}_j$ such that $\lambda_j>0$, $0<\eta_{j+1}<\delta_j<s_j<\eta_j<1$, $\lim_{j\to+\infty}\eta_j=0$, and $h(x,s,\lambda)\leqslant0$ a.e. $x\in\Omega$, for all $s\in[\delta_j,\eta_j]$, $\lambda\in[-\lambda_j,\lambda_j]$ and $j\in\mathbb{N}$ large enough.

For any $j\in\mathbb{N}$, we define $h_j(x,s,\lambda):=h(x,\tau_{\eta_j}(s),\lambda)$ and $H_j(x,s,\lambda):=\int_0^s h_j(x,t,\lambda)\,dt$, for $x\in\Omega$, $s\geqslant0$ and $\lambda\in[-\lambda_j,\lambda_j]$. By straightforward computation, we deduce that h_j satisfies all the assumptions of Lemma 2.3 for j large, with $\delta=\delta_j$ and

$\eta = \eta_j$. For any $j \in \mathbb{N}$, let $\mathcal{E}_{j,\lambda}$ be the energy functional $\mathcal{E}_{j,\lambda} := \mathcal{E}_{K,h_j(\cdot,\cdot,\lambda)}$. By Lemma 2.3, we deduce that for j sufficiently large and provided that $|\lambda| \leqslant \lambda_j$, there exists $u_{j,\lambda} \in W^{\eta_j}$ such that:

$$\min_{u \in W^{\eta_j}} \mathcal{E}_{j,\lambda}(u) = \mathcal{E}_{j,\lambda}(u_{j,\lambda}) \tag{11}$$

$$u_{j,\lambda}(x) \in [0, \delta_j] \quad \text{for a.e. } x \in \Omega, \tag{12}$$

and

$$u_{j,\lambda} \text{ is a non-negative weak solution of (5) with } h = h_j. \tag{13}$$

Since for j sufficiently large, $0 \leqslant u_{j,\lambda}(x) \leqslant \delta_j < \eta_j$ a.e. $x \in \Omega$, we have $h_j(x, u_{j,\lambda}(x), \lambda) = h(x, u_{j,\lambda}(x), \lambda)$, so that $u_{j,\lambda}$ is a non-negative weak solution of problem (1), provided that j is large and $|\lambda| \leqslant \lambda_j$.

It remains to prove that for any $k \in \mathbb{N}$, problem (1) admits at least k distinct solutions for suitable values of λ. At this purpose, we note that for any $u \in W_0^{1,p}(\Omega)$:

$$\begin{aligned}\mathcal{E}_{j,\lambda}(u) &= \int_\Omega a\big(x, \nabla u(x)\big)\,\mathrm{d}x - \frac{\lambda}{q+1}\int_\Omega \beta(x)\big|u(x)\big|^{q+1}\,\mathrm{d}x - \int_\Omega F\big(u(x)\big)\,\mathrm{d}x \\ &= \mathcal{E}_{j,0}(u) - \frac{\lambda}{q+1}\int_\Omega \beta(x)\big|u(x)\big|^{q+1}\,\mathrm{d}x.\end{aligned}$$

Claim. There exists an increasing sequence $\{\theta_j\}_j$ such that $\theta_j < 0$, $\lim_{j\to+\infty}\theta_j = 0$ and $\theta_{j-1} < \mathcal{E}_{j,0}(u_{j,0}) < \theta_j$ for $j \geqslant j^*$, with $j^* \in \mathbb{N}$.

First, note that the function $(x,s) \mapsto h(x,s,0) = \tilde{\lambda}_0 s^{p-1} + f(s)$ verifies all the assumptions of Lemma 2.2. Thus, there exist $\ell > 0$ and $\zeta \in (0, \eta_1)$ such that $F(s) \geqslant -\ell s^p$ for all $s \in (0,\zeta)$ and there is a sequence $\{\tilde{s}_j\}_j$ such that $0 < \tilde{s}_j \to 0$ as $j \to +\infty$ such that for all $L > 0$, $F(s_j) > L s_j^p$ for $j \in \mathbb{N}$ large enough. Also, since $\delta_j \searrow 0$ as $j \to +\infty$, we can choose a subsequence of $\{\delta_j\}_j$, still denoted by $\{\delta_j\}_j$, such that $\tilde{s}_j \leqslant \delta_j$ for all $j \in \mathbb{N}$.

Now, for any $s > 0$ we need to define the function z_s as follows:

$$z_s(x) := \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, r) \\ \frac{2s}{r}(r - |x - x_0|) & \text{if } x \in B(x_0, r) \setminus B(x_0, r/2) \\ s & \text{if } x \in B(x_0, r/2), \end{cases} \tag{14}$$

which is such that $z_s \geqslant 0$ in Ω, $z_s \in W_0^{1,p}(\Omega)$ and $\|z_s\|_{L^\infty(\Omega)} = s$. Here $x_0 \in \Omega$ and $r > 0$ is such that $B(x_0, r) \subset \Omega$. In the following, we denote: $\tilde{z}_j := z_{\tilde{s}_j}$.

Now, let us fix $j \in \mathbb{N}$ sufficiently large. We have $\mathcal{E}_{j,0}(u_{j,0}) \leqslant \mathcal{E}_{j,0}(\tilde{z}_j) < 0$ and

$$\mathcal{E}_{j,0}(u_{j,0}) \geqslant -\int_\Omega F\big(u_{j,0}(x)\big)\,\mathrm{d}x \geqslant -\int_\Omega \int_0^{u_{j,0}(x)} \big|f(s)\big|\,\mathrm{d}s\,\mathrm{d}x \geqslant -\int_\Omega \int_0^{\delta_j} \big|f(s)\big|\,\mathrm{d}s\,\mathrm{d}x \geqslant d_j.$$

Note that $\{c_j\}_j$ and $\{d_j\}_j$ are such that $d_j < c_j < 0$ for any $j \in \mathbb{N}$ and $\lim_{j\to+\infty} c_j = \lim_{j\to+\infty} d_j = 0$. Thus, we can extract two subsequences, still denoted by $\{c_j\}_j$ and $\{d_j\}_j$, such that the above properties hold true and the sequences $\{c_j\}_j$ and $\{d_j\}_j$ are non-decreasing. Now, we define:

$$\theta_j := \begin{cases} c_j & \text{if } j \in \mathbb{N} \text{ is even} \\ d_j & \text{if } j \in \mathbb{N} \text{ is odd.} \end{cases}$$

We deduce that for i large enough $\theta_{2i-1} = d_{2i-1} \leqslant d_{2i} < \mathcal{E}_{2i,0}(u_{2i,0}) < c_{2i} = \theta_{2i}$, which proves the claim.

Now, for any $j \geqslant j^*$, let:

$$\lambda'_j := \frac{(q+1)(\mathcal{E}_{j,0}(u_{j,0}) - \theta_{j-1})}{(\|\beta\|_{L^\infty(\Omega)} + 1)\mathcal{L}(\Omega)}, \qquad \lambda''_j := \frac{(q+1)(\theta_j - \mathcal{E}_{j,0}(u_{j,0}))}{\|\beta\|_{L^1(\Omega)} + 1}. \tag{15}$$

Note that λ'_j and λ''_j are strictly positive and they are independent of λ. For any fixed $k \in \mathbb{N}$, let:

$$\Lambda_k := \min\big\{\lambda_{j^*+1}, \dots, \lambda_{j^*+k}, \lambda'_{j^*+1}, \dots, \lambda'_{j^*+k}, \lambda''_{j^*+1}, \dots, \lambda''_{j^*+k}\big\}.$$

Of course, $\Lambda_k > 0$ is independent of λ. Also, if $|\lambda| \leqslant \Lambda_k$, then $|\lambda| \leqslant \lambda_j$ for any $j = j^*+1, \dots, j^*+k$. As a consequence of this, for any $\lambda \in \mathbb{R}$ with $|\lambda| \leqslant \Lambda_k$, $u_{j,\lambda}$ is a non-negative weak solution of problem (1) for any $j = j^*+1, \dots, j^*+k$. Let us show that these solutions are distinct. At this purpose, note that $u_{j,\lambda} \in W^{\eta_j}$ and so $\mathcal{E}_{j,0}(u_{j,0}) = \min_{u \in W^{\eta_j}} \mathcal{E}_{j,0}(u) \leqslant \mathcal{E}_{j,0}(u_{j,\lambda})$. Thus, for any λ such that $|\lambda| \leqslant \Lambda_k$, we obtain:

$$\begin{aligned}\mathcal{E}_{j,\lambda}(u_{j,\lambda}) &= \mathcal{E}_{j,0}(u_{j,\lambda}) - \frac{\lambda}{q+1}\int_{\Omega} \beta(x)\left|u_{j,\lambda}(x)\right|^{q+1} \mathrm{d}x \geqslant \mathcal{E}_{j,0}(u_{j,0}) - \frac{|\lambda|}{q+1}\|\beta\|_{L^\infty(\Omega)}\eta_j^{q+1}\mathcal{L}(\Omega) \\ &\geqslant \mathcal{E}_{j,0}(u_{j,0}) - \frac{\Lambda_k}{q+1}\|\beta\|_{L^\infty(\Omega)}\mathcal{L}(\Omega) \geqslant \mathcal{E}_{j,0}(u_{j,0}) - \frac{\lambda'_j}{q+1}\|\beta\|_{L^\infty(\Omega)}\mathcal{L}(\Omega) = \theta_{j-1},\end{aligned} \tag{16}$$

for any $j = j^*+1, \ldots, j^*+k$. On the other hand, using the fact that $\|\tilde{z}_j\|_{L^\infty(\Omega)} = \tilde{s}_j \leqslant \delta_j < \eta_j < 1$, for any λ with $|\lambda| \leqslant \Lambda_k$ we deduce that:

$$\begin{aligned}\mathcal{E}_{j,\lambda}(u_{j,\lambda}) &= \min_{u \in W^{\eta_j}} \mathcal{E}_{j,\lambda}(u) \leqslant \mathcal{E}_{j,\lambda}(\tilde{z}_j) = \mathcal{E}_{j,0}(\tilde{z}_j) - \frac{\lambda}{q+1}\int_{\Omega} \beta(x)\left|\tilde{z}_j(x)\right|^{q+1} \mathrm{d}x \\ &\leqslant \mathcal{E}_{j,0}(\tilde{z}_j) - \frac{\lambda}{q+1}\int_{\{x\in\Omega:\ \lambda\beta(x)<0\}} \beta(x)\left|\tilde{z}_j(x)\right|^{q+1} \mathrm{d}x \leqslant \mathcal{E}_{j,0}(\tilde{z}_j) - \frac{\lambda}{q+1}\int_{\{x\in\Omega:\ \lambda\beta(x)<0\}} \beta(x)\,\mathrm{d}x \\ &\leqslant \mathcal{E}_{j,0}(\tilde{z}_j) + \frac{|\lambda|}{q+1}\int_{\{x\in\Omega:\ \lambda\beta(x)<0\}} \left|\beta(x)\right| \mathrm{d}x \leqslant \mathcal{E}_{j,0}(\tilde{z}_j) + \frac{\lambda}{q+1}\int_{\Omega}\left|\beta(x)\right| \mathrm{d}x \\ &\leqslant \mathcal{E}_{j,0}(\tilde{z}_j) + \frac{\Lambda_k}{q+1}\|\beta\|_{L^1(\Omega)} \leqslant \mathcal{E}_{j,0}(\tilde{z}_j) + \frac{\lambda''_j}{q+1}\|\beta\|_{L^1(\Omega)} = \theta_j\end{aligned} \tag{17}$$

for any $j = j^*+1, \ldots, j^*+k$.

Hence, by (16), (17) and the properties of $\{\theta_j\}_j$, we deduce that for any $j = j^*+1, \ldots, j^*+k$

$$\theta_{j-1} < \mathcal{E}_{j,\lambda}(u_{j,\lambda}) < \theta_j < 0, \tag{18}$$

which yields that $\mathcal{E}_{1,\lambda}(u_{1,\lambda}) < \cdots < \mathcal{E}_{k,\lambda}(u_{k,\lambda}) < 0$. Thus, the solutions $\{u_{1,\lambda}, \ldots, u_{k,\lambda}\}$ are all distinct and non-trivial, provided that $|\lambda| \leqslant \Lambda_k$.

Finally, we estimate the $W_0^{1,p}$-norm of $u_{j,\lambda}$. For all $j = j^*+1, \ldots, j^*+k$ and $|\lambda| \leqslant \Lambda_k$, we have:

$$\begin{aligned}\frac{\Gamma_1}{p}\|u_{j,\lambda}\|^p_{W_0^{1,p}(\Omega)} &\leqslant \mathcal{E}_{j,\lambda}(u_{j,\lambda}) + \frac{\lambda}{q+1}\int_{\Omega} \beta(x)\left|u_{j,\lambda}(x)\right|^{q+1} \mathrm{d}x + \int_{\Omega} F\left(u_{j,\lambda}(x)\right) \mathrm{d}x \\ &< \theta_j + \frac{|\lambda|}{q+1}\|\beta\|_{L^\infty(\Omega)}\delta_j^{q+1} + \int_{\Omega}\int_0^{\delta_j}\left|f(s)\right| \mathrm{d}s\,\mathrm{d}x < \frac{\Lambda_k}{q+1}\|\beta\|_{L^\infty(\Omega)}\delta_j + \bar{C}\delta_j,\end{aligned}$$

for a suitable positive constant $\bar{C}$. It follows that $\|u_{j,\lambda}\|_{W_0^{1,p}(\Omega)} \leqslant \tilde{C}\delta_j^{1/p}$, where $\tilde{C} > 0$. Since $\delta_j \to 0$ as $j \to +\infty$, without loss of generality, we may assume that $\delta_j \leqslant \min\{\tilde{C}^{-p}, 1\}1/j^p$, and this gives $\|u_{j,\lambda}\|_{W_0^{1,p}(\Omega)} \leqslant 1/j$ for all $j = j^*+1, \ldots, j^*+k$, provided that $|\lambda| \leqslant \Lambda_k$. This completes the proof of Theorem 2.1. □

3. Oscillation at infinity

In this section, we assume that the nonlinear term f satisfies the following assumptions:

$$\liminf_{s\to+\infty} \frac{f(s)}{s^{p-1}} =: -\ell_\infty \in [-\infty, 0); \tag{19}$$

$$-\infty < \liminf_{s\to+\infty} \frac{F(s)}{s^p} \leqslant \limsup_{s\to+\infty} \frac{F(s)}{s^p} = +\infty. \tag{20}$$

A function satisfying these conditions is $f(s) = \alpha s^{\alpha-1}(1-\sin s^\sigma) - \sigma s^{\alpha+\sigma-1}\cos s^\sigma - p\gamma s^{p-1}$, where α, σ and γ are such that $\alpha > p$, $\sigma > 0$ and $\gamma > 0$.

In this setting, the counterpart of Theorem 2.1 can be stated as follows.

Theorem 3.1. *Assume that f satisfies relations (19), (20), and $f(0) = 0$. If either*

a) *$q = p-1$, $\ell_\infty \in (0, +\infty)$ and $\lambda\beta(x) < \lambda_\infty$ a.e. $x \in \Omega$ for some $\lambda_\infty \in (0, \ell_\infty)$ or*

b) $q = p - 1$, $\ell_\infty = +\infty$ *and* $\lambda \in \mathbb{R}$ *is arbitrary or*
c) $0 < q < p - 1$ *and* $\lambda \in \mathbb{R}$ *is arbitrary,*

then there exists a sequence $\{u_j\}_j$ *in* $W_0^{1,p}(\Omega)$ *of distinct weak solutions of problem* (1) *such that*

$$\lim_{j \to +\infty} \|u_j\|_{L^\infty(\Omega)} = +\infty.$$

Assume that $q > p - 1$. *Then for every* $k \in \mathbb{N}$ *there exists* $\Lambda_k > 0$ *such that problem* (1) *has at least* k *distinct weak solutions* $u_1, \ldots, u_k \in W_0^{1,p}(\Omega)$ *satisfying* $\|u_j\|_{L^\infty(\Omega)} \geqslant j - 1$ *for all* $j = 1, \ldots, k$, *provided that* $|\lambda| < \Lambda_k$.

We refer to [4] for the proof and several related results. We also refer to the marvelous recent book by Ciarlet [2] for the rigorous qualitative analysis of many models described by nonlinear partial differential equations.

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