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Dynamical systems

The resurgent character of the Fatou coordinates of a simple parabolic germ



Résurgence des coordonnées de Fatou des germes paraboliques simples

Artem Dudko^a, David Sauzin^b

^a Institute for Mathematical Sciences, University of Stony Brook, NY, USA

^b CNRS UMI 3483, Laboratoire Fibonacci, Centro di Ricerca Matematica Ennio De Giorgi, Scuola Normale Superiore di Pisa, Italy

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ABSTRACT

Given a holomorphic germ at the origin of \mathbb{C} with a simple parabolic fixed point, the local dynamics is classically described by means of pairs of attracting and repelling Fatou coordinates and the corresponding pairs of horn maps, of crucial importance for Écalle–Voronin's classification result and the definition of the parabolic renormalization operator. We revisit Écalle's approach to the construction of Fatou coordinates, which relies on Borel–Laplace summation, and give an original and self-contained proof of their resurgent character.

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RÉSUMÉ

Pour un germe holomorphe à l'origine de \mathbb{C} avec un point fixe parabolique simple, la dynamique locale se décrit classiquement à l'aide d'une paire de coordonnées de Fatou, qui permet de définir une paire d'applications de corne, cruciale pour le résultat de classification analytique d'Écalle–Voronin et pour la définition de l'opérateur de renormalisation parabolique. Nous revisitons l'approche d'Écalle fondée sur la sommation de Borel–Laplace pour construire les coordonnées de Fatou et donnons une démonstration originale complète de leur caractère résurgent.

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Version française abrégée

Nous nous donnons un germe holomorphe parabolique simple dont nous plaçons le point fixe à l'infini ; il se met alors sous la forme $f(z) = z + 1 + a(z)$, avec $a(z) \in z^{-1}\mathbb{C}\{z^{-1}\}$. Nous notons ρ l'opposé du coefficient de z^{-1} dans $a(z)$. Pour trouver des paires (v^+, v^-) de coordonnées de Fatou, c'est-à-dire des fonctions qui conjuguent f et la translation $\text{id} + 1$ univalentes dans un domaine de la forme $\mathcal{D}_{R, \delta_0}^\pm$ défini ci-dessous, nous les cherchons sous la forme $v(z) = z + \rho \text{Log} z + \varphi(z)$ où $\text{Log} z$ est une détermination quelconque du logarithme ; il est alors équivalent de résoudre :

$$C_{\text{id}-1}\varphi = C_{\text{id}+b}\varphi + b_1,$$

où $b(z) := a(z-1) \in z^{-1}\mathbb{C}\{z^{-1}\}$, $b_1(z) := b(z) + \rho \log \frac{1+z^{-1}b(z)}{1-z^{-1}} \in z^{-2}\mathbb{C}\{z^{-1}\}$, et $C_g :=$ opérateur de composition avec g .

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Théorème 1. Il y a une unique solution formelle de la forme $\tilde{\varphi}(z) = \sum_{n \geq 0} c_n z^{-n-1}$. Sa transformée de Borel formelle $\hat{\varphi} := \mathcal{B}\tilde{\varphi} = \sum_{n \geq 0} c_n \frac{z^n}{n!}$ est $2\pi i\mathbb{Z}$ -résurgente, c'est-à-dire qu'elle converge dans $\mathbb{D}_{2\pi} := \{|\zeta| < 2\pi\}$ et admet un prolongement analytique le long de tout chemin γ issu de $\mathbb{D}_{2\pi}$ restant dans $\mathbb{C} \setminus 2\pi i\mathbb{Z}$.

De plus, pour un tel chemin γ aboutissant en $\zeta_* \in i\mathbb{R}$, ou pour $\gamma = \{0\}$ et $\zeta_* = 0$, étant donné $\delta_0 \in (0, \frac{\pi}{2})$, il existe $C_0(\delta_0), R_0(\delta_0) > 0$ tels que $\text{cont}_\gamma \hat{\varphi}$ (germe holomorphe en ζ_* obtenu par prolongement analytique le long de γ) vérifie :

$$|\text{cont}_\gamma \hat{\varphi}(\zeta_* + t e^{i\theta})| \leq C_0(\delta_0) e^{R_0(\delta_0)t} \quad \text{pour } t \geq 0 \text{ et } \theta \in I_{\delta_0}^+ \cup I_{\delta_0}^-, \tag{1}$$

avec $I_{\delta_0}^+ := [-\delta_0, \delta_0]$ et $I_{\delta_0}^- := [\pi - \delta_0, \pi + \delta_0]$.

En particulier, pour $\delta_0 \in (0, \frac{\pi}{2})$, nous avons une transformée de Laplace $\mathcal{L}^\theta \hat{\varphi}(z) := \int_0^{e^{i\theta}\infty} e^{-z\zeta} \hat{\varphi}(\zeta) d\zeta$ holomorphe dans $\Pi_{R_0(\delta_0)}^\theta := \{z \in \mathbb{C} \mid \text{Re}(z e^{i\theta}) > R_0(\delta_0)\}$ pour chaque $\theta \in I_{\delta_0}^\pm$. Toutes ces fonctions se recollent et fournissent :

$$\mathcal{L}^+ \hat{\varphi} \text{ holomorphe dans } U^+ := \bigcup_{\delta_0 \in (0, \frac{\pi}{2})} \bigcup_{\theta \in I_{\delta_0}^+} \Pi_{R_0(\delta_0)}^\theta, \quad \mathcal{L}^- \hat{\varphi} \text{ holomorphe dans } U^- := \bigcup_{\delta_0 \in (0, \frac{\pi}{2})} \bigcup_{\theta \in I_{\delta_0}^-} \Pi_{R_0(\delta_0)}^\theta.$$

Le domaine U^+ est contenu dans $\mathbb{C} \setminus \mathbb{R}^-$, où est holomorphe la détermination principale du logarithme Log^+ , alors que $U^- \subset \mathbb{C} \setminus \mathbb{R}^+$, où est holomorphe le prolongement analytique Log^- obtenu en tournant dans le sens direct. Posons :

$$\mathcal{P}_{R,\delta}^+ := \left\{ r e^{i\theta} \in \mathbb{C} \mid r > R, -\frac{\pi}{2} - \delta_0 < \theta < \frac{\pi}{2} + \delta_0 \right\}, \quad \mathcal{P}_{R,\delta}^- := \left\{ r e^{i\theta} \in \mathbb{C} \mid r > R, \frac{\pi}{2} - \delta_0 < \theta < \frac{3\pi}{2} + \delta_0 \right\}.$$

Théorème 2. Les formules $v_*^+ := \text{id} + \rho \text{Log}^+ + \mathcal{L}^+ \hat{\varphi}$ et $v_*^- := \text{id} + \rho \text{Log}^- + \mathcal{L}^- \hat{\varphi}$ définissent une paire de coordonnées de Fatou. Pour $0 < \delta' < \delta < \pi/2$ et R assez grand, $\mathcal{L}^\pm \hat{\varphi}$ admet $\tilde{\varphi}$ pour développement asymptotique 1-Gevrey uniformément dans $\mathcal{P}_{R,\delta}^\pm$, v_*^\pm est univalente dans $\mathcal{P}_{R,\delta}^\pm$ et $v_*^\pm(\mathcal{P}_{R,\delta}^\pm)$ contient un ensemble de la forme $\mathcal{P}_{R',\delta'}^\pm$.

Les autres paires de coordonnées de Fatou sont les $(v_*^+ + c^+, v_*^- + c^-)$ avec $c^+, c^- \in \mathbb{C}$ arbitraires.

C'est dans [3] qu'a été utilisée pour la première fois la sommation de Borel-Laplace pour construire des coordonnées de Fatou. On y trouve aussi l'énoncé de notre Théorème 1, avec une démonstration détaillée seulement lorsque $\rho = 0$; notre démonstration est plus simple et originale.

On définit les applications de corne relevées $h_*^{\text{up/low}}$ comme restrictions de $v_*^+ \circ (v_*^-)^{-1}$ à chacune des deux composantes connexes de $\mathcal{P}_{R,\delta}^+ \cap (v_*^-)^{-1}(\mathcal{P}_{R',\delta'}^-)$; elles se prolongent à un demi-plan supérieur/inférieur et sont de la forme :

$$h_*^{\text{up}}(Z) = Z + \sum_{m \geq 1} A_m e^{2\pi i m Z}, \quad h_*^{\text{low}}(Z) = Z - 2\pi i \rho + \sum_{m \geq 1} A_{-m} e^{-2\pi i m Z}.$$

Elles permettent de classifier analytiquement les germes paraboliques simples et c'est à partir de $f \mapsto h_*^{\text{up}}$ que l'on définit l'opérateur de renormalisation parabolique, si important dans les travaux récents de dynamique holomorphe.

1. Resurgence and summability of the formal Fatou coordinate

Let us give ourselves a simple parabolic germ of holomorphic map at the origin of \mathbb{C} , i.e. of the form:

$$F(w) = w + cw^2 + O(w^3) \in w\mathbb{C}\{w\} \tag{2}$$

with $c \in \mathbb{C}^*$. We rewrite it at ∞ by means of the change of coordinate $z = -1/cw$:

$$f(z) := -\frac{1}{cF(-\frac{1}{cz})} = z + 1 + a(z), \quad a(z) = -\rho z^{-1} + O(z^{-2}) \in z^{-1}\mathbb{C}\{z^{-1}\} \tag{3}$$

with some $\rho \in \mathbb{C}$. For every $R > 0$ and $\delta \in (0, \pi/2)$, we introduce the notations:

$$\mathcal{P}_{R,\delta}^+ := \left\{ r e^{i\theta} \in \mathbb{C} \mid r > R, -\frac{\pi}{2} - \delta < \theta < \frac{\pi}{2} + \delta \right\}, \quad \mathcal{P}_{R,\delta}^- := \left\{ r e^{i\theta} \in \mathbb{C} \mid r > R, \frac{\pi}{2} - \delta < \theta < \frac{3\pi}{2} + \delta \right\}.$$

Definition 1. A pair of Fatou coordinates at ∞ is a pair (v^+, v^-) of univalent maps:

$$v^+ : \mathcal{P}_{R,\delta}^+ \rightarrow \mathbb{C}, \quad v^- : \mathcal{P}_{R,\delta}^- \rightarrow \mathbb{C},$$

for some R and δ , such that:

$$v^+ \circ f = v^+ + 1, \quad v^- \circ f = v^- + 1. \tag{4}$$

It is well known that pairs of Fatou coordinates exist and are unique up to the addition of a free pair of constants $(c^+, c^-) \in \mathbb{C}^2$. We shall see how to recover these facts by means of the Borel–Laplace summation.

Notation. Let $b(z) := a(z - 1)$ and $b_1(z) := b(z) + \rho \log \frac{1+z^{-1}b(z)}{1-z^{-1}}$, so that $b(z) \in z^{-1}\mathbb{C}\{z^{-1}\}$ and $b_1(z) \in z^{-2}\mathbb{C}\{z^{-1}\}$. We denote by $C_{\text{id}-1}$ and $C_{\text{id}+b}$ the composition operators $\varphi(z) \mapsto \varphi(z - 1)$ and $\varphi(z) \mapsto \varphi(z + b(z))$, acting in spaces of functions as well as in the space of formal series $\mathbb{C}[[z^{-1}]]$.

Suppose that $\mathcal{P} = \mathcal{P}_{R,\delta}^\pm$ with R large enough so that b is analytic on \mathcal{P} and let Log be any branch of the logarithm in \mathcal{P} . Under the change of unknown function $v(z) = z + \rho \text{Log} z + \varphi(z)$, the equation $v \circ f = v + 1$ is transformed¹ into:

$$C_{\text{id}-1}\varphi = C_{\text{id}+b}\varphi + b_1. \tag{5}$$

Theorem 1. Eq. (5) admits a unique formal solution of the form $\tilde{\varphi}(z) = \sum_{n \geq 0} c_n z^{-n-1}$. Its formal Borel transform $\hat{\varphi} := \mathcal{B}\tilde{\varphi} = \sum_{n \geq 0} c_n \frac{z^n}{n!}$ is $2\pi i\mathbb{Z}$ -resurgent, in the sense that it converges for $|\zeta| < 2\pi$ and extends analytically along any path issuing from 0 and staying in $\mathbb{C} \setminus 2\pi i\mathbb{Z}$ except for its origin.

Moreover, for such a path γ with endpoint $\zeta_* \in i\mathbb{R}$ or for $\gamma = \{0\}$ and $\zeta_* = 0$, and for $\delta_0 \in (0, \frac{\pi}{2})$, there exist $C_0, R_0 > 0$ such that the analytic continuation $\text{cont}_\gamma \hat{\varphi}$ of $\hat{\varphi}$ along γ (which is a holomorphic germ at ζ_*) satisfies:

$$|\text{cont}_\gamma \hat{\varphi}(\zeta_* + t e^{i\theta})| \leq C_0 e^{R_0 t} \quad \text{for all } t \geq 0 \text{ and } \theta \in I_{\delta_0}^+ \cup I_{\delta_0}^-, \tag{6}$$

where $I_{\delta_0}^+ := [-\delta_0, \delta_0]$ and $I_{\delta_0}^- := [\pi - \delta_0, \pi + \delta_0]$.

In the case $\gamma = \{0\}$, inequality (6) says that the formal series $\tilde{\varphi}$ is Borel–Laplace 1-summable in the directions of $I_{\delta_0}^+$ and $I_{\delta_0}^-$. In fact, Theorem 1 says much more, since it provides a Riemann surface to which $\hat{\varphi}$ extends, with exponential estimates on all its sheets. The rest of this section is devoted to the proof of Theorem 1.

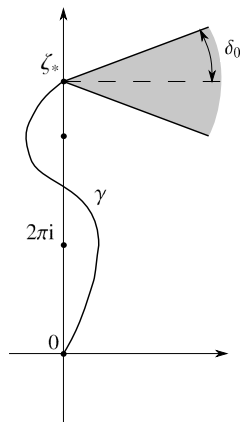
The operator $C_{\text{id}-1} - \text{Id}$ maps $z^{-1}\mathbb{C}[[z^{-1}]]$ to $z^{-2}\mathbb{C}[[z^{-1}]]$ bijectively; we denote by E the inverse, which decreases the order by one unit and has as Borel counterpart the multiplication operator:

$$\hat{E} : \hat{\psi}(\zeta) \in \zeta \mathbb{C}[[\zeta]] \mapsto \frac{1}{e^\zeta - 1} \hat{\psi}(\zeta) \in \mathbb{C}[[\zeta]].$$

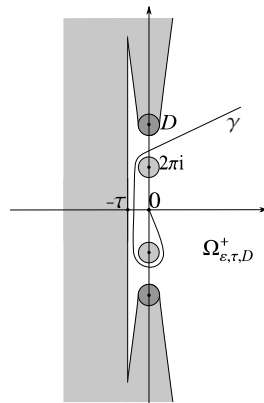
On the other hand, $B := C_{\text{id}+b} - \text{Id} = \sum_{k \geq 1} \frac{1}{k!} b^k (\frac{d}{dz})^k$ (formally convergent series of operators) increases the order by at least two units, so that the composition EB maps $z^{-k-1}\mathbb{C}[[z^{-1}]]$ in $z^{-k-2}\mathbb{C}[[z^{-1}]]$ for every $k \geq 0$.

Lemma 1. Let $\tilde{\varphi}_k := (EB)^k E b_1 \in z^{-k-1}\mathbb{C}[[z^{-1}]]$ for $k \geq 0$. Then the formally convergent series $\sum_{k \geq 0} \tilde{\varphi}_k$ is the unique solution of (5) in $z^{-1}\mathbb{C}[[z^{-1}]]$.

Proof. For $\tilde{\varphi} \in z^{-1}\mathbb{C}[[z^{-1}]]$, (5) $\Leftrightarrow (C_{\text{id}-1} - \text{Id})\tilde{\varphi} = B\tilde{\varphi} + b_1 \Leftrightarrow \tilde{\varphi} = EB\tilde{\varphi} + E b_1$, whence the conclusion follows. \square



(a) Illustration of Theorem 1.



(b) Illustration of Lemma 2.

¹ Notice that, for $|z|$ large enough, $\frac{f(z)}{z}$ is close to 1, hence $\text{Log} f(z) - \text{Log} z = \log \frac{f(z)}{z}$ where \log is the principal branch of the logarithm.

Lemma 2. For each $k \geq 0$ the formal Borel transform $\hat{\varphi}_k := \mathcal{B}\tilde{\varphi}_k$ is $2\pi i\mathbb{Z}$ -resurgent.

Suppose $0 < \varepsilon < \pi < \tau$, $0 < \kappa \leq 1$ and D is a closed disc of radius ε centred at $2\pi im_0$ with $m_0 \in \mathbb{Z}^*$. Then there exist $C_1, M, R > 0$ such that, for every piecewise C^1 path with natural parameterisation $\gamma : [0, \ell] \rightarrow \mathbb{C}$ satisfying:

$$s \in [0, \varepsilon] \Rightarrow |\gamma(s)| = s, \quad s > \varepsilon \Rightarrow |\gamma(s)| > \varepsilon, \quad s \in [0, \ell] \Rightarrow |\gamma(s)| > \kappa s, \tag{7}$$

$$\gamma([0, \ell]) \subset \Omega_{\varepsilon, \tau, D}^+, \tag{8}$$

where $\Omega_{\varepsilon, \tau, D}^+ := \{\zeta \in \mathbb{C} \mid \operatorname{Re} \zeta > -\tau, \operatorname{dist}(\zeta, 2\pi i\mathbb{Z}^*) > \varepsilon\} \setminus \{u\zeta \in \mathbb{C} \mid u \in [1, +\infty), \pm \zeta \in D\}$,

one has:

$$|\operatorname{cont}_\gamma \hat{\varphi}_k(\gamma(\ell))| \leq C_1 \frac{(M\ell)^k}{k!} e^{R\ell} \quad \text{for all } k \geq 0. \tag{9}$$

Proof. Choose $M_0, \lambda > 0$ so that $|\frac{\zeta}{e^\zeta - 1}| \leq M_0 e^{-\lambda|\zeta|}$ for $\zeta \in \Omega_{\varepsilon, \tau, D}^+$. We have $\hat{b} := \mathcal{B}b$, $\hat{b}_1 := \mathcal{B}b_1$ entire functions, with

$$|\hat{b}(\zeta)| \leq C e^{\beta|\zeta|} \quad \text{and} \quad |\hat{b}_1(\zeta)| \leq C|\zeta| e^{\beta|\zeta|} \quad \text{for all } \zeta \in \mathbb{C},$$

for some $C, \beta > 0$. The formal Borel transform maps $b^k(z)$ to $\hat{b}^{*k}(\zeta)$, the k -th convolution² power of $\hat{b}(\zeta)$, which is also entire and for which we get:

$$|\hat{b}^{*k}(\zeta)| \leq C^k \frac{|\zeta|^{k-1}}{(k-1)!} e^{\beta|\zeta|} \quad \text{for all } \zeta \in \mathbb{C}. \tag{10}$$

The Borel counterpart of $C_{\operatorname{id}+b} - \operatorname{Id}$ is $\hat{B} : \hat{\psi} \in \mathbb{C}[[\zeta]] \mapsto \sum_{k \geq 1} \frac{1}{k!} \hat{b}^{*k} * ((-\zeta)^k \hat{\psi}) \in \mathbb{C}[[\zeta]]$, which leaves invariant the subspace $\mathbb{C}\{\zeta\}$ and induces on it an integral transform:

$$\hat{B}\hat{\psi}(\zeta) = \int_0^\zeta K(\xi, \zeta) \hat{\psi}(\xi) d\xi \quad \text{for } \zeta \text{ close to } 0, \quad \text{with kernel function } K(\xi, \zeta) = \sum_{k \geq 1} \frac{(-\xi)^k}{k!} \hat{b}^{*k}(\zeta - \xi).$$

The point is that this kernel is holomorphic in $\mathbb{C} \times \mathbb{C}$, with estimates following from (10):

$$|K(\xi, \zeta)| \leq C|\xi| e^{\mu|\xi| + (\frac{C}{\mu} + \beta)|\zeta - \xi|} \quad \text{for all } (\xi, \zeta) \in \mathbb{C} \times \mathbb{C},$$

with any $\mu > 0$. Therefore, whenever a germ $\hat{\psi} \in \mathbb{C}\{\zeta\}$ admits an analytic continuation along a naturally parameterised path $\gamma : [0, \ell] \rightarrow \mathbb{C}$, this is also true for $\hat{B}\hat{\psi}$, with $\operatorname{cont}_\gamma \hat{B}\hat{\psi}(\gamma(s)) = \int_0^s K(\gamma(\sigma), \gamma(s)) \operatorname{cont}_\gamma \hat{\psi}(\gamma(\sigma)) \gamma'(\sigma) d\sigma$ for all $s \in [0, \ell]$. Hence $2\pi i\mathbb{Z}$ -resurgence is preserved by the composition $\hat{E}\hat{B}$ and we can get estimates.

We start with $\hat{\varphi}_0 = \hat{E}\hat{b}_1 = \frac{\hat{b}_1(\zeta)}{e^\zeta - 1}$, which is meromorphic and clearly $2\pi i\mathbb{Z}$ -resurgent, and satisfies (9) with $C_1 := M_0 C$ and with any $R \geq \beta$. We choose $\mu := \lambda\kappa$, $R := \frac{C}{\mu} + \beta$ and $M := \frac{M_0 C}{\kappa}$. We obtain the desired result for $\hat{\varphi}_k = (\hat{E}\hat{B})^k \hat{\varphi}_0$ by induction, observing that if a path γ satisfies (7)–(8) and a $2\pi i\mathbb{Z}$ -resurgent function $\hat{\psi}$ satisfies $|\operatorname{cont}_\gamma \hat{\psi}(\gamma(s))| \leq e^{R_s} \Psi(s)$ for all $s \in [0, \ell]$ then, for all $0 \leq \sigma \leq s \leq \ell$, $|K(\gamma(\sigma), \gamma(s))| \leq C\sigma e^{\mu\sigma + R(s-\sigma)}$ and $|\frac{1}{e^{\gamma(s)} - 1}| \leq \frac{M_0}{\kappa s} e^{-\mu s}$, whence:

$$|\operatorname{cont}_\gamma \hat{E}\hat{B}\hat{\psi}(\gamma(s))| \leq \frac{CM_0}{\kappa} e^{R_s} \int_0^s \frac{\sigma}{s} \Psi(\sigma) e^{-\mu(s-\sigma)} d\sigma \leq M e^{R_s} \int_0^s \Psi(\sigma) d\sigma \quad \text{for all } s \in [0, \ell]. \quad \square$$

We can deduce that the series of holomorphic functions $\sum_{k \geq 0} \hat{\varphi}_k$ converges normally in any compact subset of the disc $\{|\zeta| < 2\pi\}$ (using paths γ of the form $[0, \zeta]$) and that its sum, which is $\hat{\varphi}$, extends analytically along any naturally parameterised path γ which starts as the line segment $[0, 1]$ and then stays in $\mathbb{C} \setminus 2\pi i\mathbb{Z}$: indeed, taking ε, κ small enough and τ, m_0 large enough, we see that Lemma 2 applies to γ and the neighbouring paths, so that (9) yields the normal convergence of $\sum_{k \geq 0} \operatorname{cont}_\gamma \hat{\varphi}_k(\gamma(t) + \zeta) = \operatorname{cont}_\gamma \hat{\varphi}(\gamma(t) + \zeta)$ for all t and ζ with $|\zeta|$ small enough. Therefore $\hat{\varphi}$ is $2\pi i\mathbb{Z}$ -resurgent.

We also get the part of (6) relative to $I_{\delta_0}^+$: given $\delta_0 \in (0, \pi/2)$ and γ with endpoint $\zeta_* \in i\mathbb{R}$, we first replace an initial portion of γ with a line segment of length 1 (unless γ stays in the unit disc, in which case the modification of the arguments which follow is trivial) and switch to its natural parameterisation $\gamma : [0, \ell] \rightarrow \mathbb{C}$. We then choose:

² The formal Borel transform \mathcal{B} maps the Cauchy product of formal series to the convolution product: $\hat{\phi} * \hat{\psi}(\zeta) = \int_0^\zeta \hat{\phi}(\zeta - \xi) \hat{\psi}(\xi) d\xi$, with termwise integration for formal series $\hat{\phi}, \hat{\psi} \in \mathbb{C}[[\zeta]]$, and with the obvious analytical meaning when $\hat{\phi}, \hat{\psi} \in \mathbb{C}\{\zeta\}$ (then taking ζ close enough to 0).

$$\varepsilon < \min \left\{ 1, \min_{[1, \ell]} |\gamma|, \text{dist}(\gamma([0, \ell]), 2\pi i\mathbb{Z}^*), \text{dist}(\zeta_*, 2\pi i\mathbb{Z}) \cos \delta_0 \right\}, \quad \kappa < \min \left\{ \min_{[0, \ell]} \frac{|\gamma(s)|}{s}, \min_{t \geq 0} \frac{|\zeta_* + t e^{\pm i\delta_0}|}{\ell + t} \right\},$$

$\tau > -\min \text{Re } \gamma$ and $m_0 > \frac{1}{2\pi}(\varepsilon + \max |\text{Im } \gamma|)$, so that, for each $t \geq 0$ and $\theta \in I_{\delta_0}^+$, **Lemma 2** applies to the concatenation $\Gamma := \gamma + [\zeta_*, \zeta_* + t e^{i\theta}]$; since Γ has length $\ell + t$, (9) yields $|\text{cont}_\gamma \hat{\varphi}(\zeta_* + t e^{i\theta})| = |\text{cont}_\Gamma \hat{\varphi}(\Gamma(\ell + t))| \leq C_1 e^{(M+R)(\ell+t)}$.

The part of (6) relative to $I_{\delta_0}^-$ follows from the fact that $\hat{\varphi}^-(\zeta) := \hat{\varphi}(-\zeta)$ satisfies all the properties we just obtained for $\hat{\varphi}(\zeta)$, since it is the formal Borel transform of $\tilde{\varphi}^-(z) := -\tilde{\varphi}(-z)$ which solves the equation $C_{\text{id}-1} \tilde{\varphi}^- = C_{\text{id}+b^-} \tilde{\varphi}^- + b_1^-$ associated with the simple parabolic germ $f^-(z) := -f^{-1}(-z) = z + 1 + b^-(z + 1)$.

The proof of **Theorem 1** is now complete.

Remark 1. The name “resurgent”, whose definition in a particular case is given in the statement of **Theorem 1**, was chosen by Écalle because of the explicit connection he obtained between the germ $\hat{\varphi}$ at 0 and the singularities of its analytic continuation at any point of $2\pi i\mathbb{Z}$ – see [1]. **Theorem 1** is stated in [3], with a detailed proof only for the case $\rho = 0$. The above proof is original and simpler.

Remark 2. For a path γ ending at ζ , with the notation $\Delta_{\gamma,k} := \{(\gamma(s_1), \dots, \gamma(s_k)) \mid s_1 \leq \dots \leq s_k\}$ for each positive integer k , we get the explicit formula $\text{cont}_\gamma \hat{\varphi}(\zeta) = \frac{1}{e^{\zeta}-1} (\hat{b}_1(\zeta) + \sum_{k \geq 1} \int_{\Delta_{\gamma,k}} \hat{b}_1(\xi_1) \frac{K(\xi_1, \xi_2) \dots K(\xi_{k-1}, \xi_k) K(\xi_k, \zeta)}{(e^{\xi_1}-1) \dots (e^{\xi_{k-1}}-1)(e^{\xi_k}-1)} d\xi_1 \wedge \dots \wedge d\xi_k)$.

2. Fatou coordinates and lifted horn maps

For $\gamma = \{0\}$ and any $\delta_0 \in (0, \pi/2)$, **Theorem 1** provides $R_0 = R_0(\delta_0)$ and we may apply the Laplace transform:

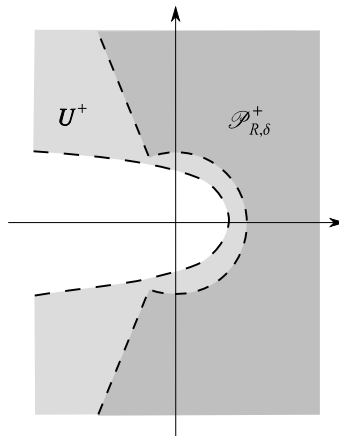
$$\mathcal{L}^\theta \hat{\varphi}(z) := \int_0^{e^{i\theta}\infty} e^{-z\zeta} \hat{\varphi}(\zeta) d\zeta, \quad \mathcal{L}^\theta \hat{\varphi} \text{ holomorphic in } \Pi_{R_0(\delta_0)}^\theta := \{z \in \mathbb{C} \mid \text{Re}(ze^{i\theta}) > R_0(\delta_0)\}, \quad \theta \in I_{\delta_0}^+ \cup I_{\delta_0}^-.$$

The functions $\mathcal{L}^\theta \hat{\varphi}$ with $\theta \in I_{\delta_0}^+$ are extension of each other, similarly for $I_{\delta_0}^-$, so we get two Borel–Laplace sums:

$$\mathcal{L}^+ \hat{\varphi} \text{ holomorphic in } U^+ := \bigcup_{\delta_0 \in (0, \frac{\pi}{2})} \bigcup_{\theta \in I_{\delta_0}^+} \Pi_{R_0(\delta_0)}^\theta, \quad \mathcal{L}^- \hat{\varphi} \text{ holomorphic in } U^- := \bigcup_{\delta_0 \in (0, \frac{\pi}{2})} \bigcup_{\theta \in I_{\delta_0}^-} \Pi_{R_0(\delta_0)}^\theta.$$

Since $U^\pm \subset \mathbb{C} \setminus \mathbb{R}^\mp$, we choose branches of the logarithm as follows:

$$\text{Log}^+ z = \int_1^z \frac{du}{u} \quad \text{for } z \in \mathbb{C} \setminus \mathbb{R}^-, \quad \text{Log}^- z = i\pi + \int_{-1}^z \frac{du}{u} \quad \text{for } z \in \mathbb{C} \setminus \mathbb{R}^+.$$



(c) Illustration of Theorem 2.

Theorem 2. The formulas:

$$v_*^\pm(z) := z + \rho \text{Log}^\pm z + \mathcal{L}^\pm \hat{\varphi}(z), \quad z \in U^\pm, \tag{11}$$

define a pair (v_*^+, v_*^-) of Fatou coordinates at ∞ for the simple parabolic germ f . For any $0 < \delta' < \delta < \pi/2$, provided that R is large enough, the function $\mathcal{L}^\pm \hat{\varphi}$ has uniform 1-Gevrey asymptotic expansion $\tilde{\varphi}$ in $\mathcal{D}_{R,\delta}^\pm$ and the function v_*^\pm is univalent in $\mathcal{D}_{R,\delta}^\pm$ with its image containing a set of the form $\mathcal{D}_{R',\delta'}^\pm$.

Any other pair of Fatou coordinates is of the form $(v_*^+ + c^+, v_*^- + c^-)$ with arbitrary complex constants c^+, c^- .

(For the definition and properties of 1-Gevrey asymptotic expansion, see [6] or [7].)

Proof. Since b_1 is convergent, it coincides with $\mathcal{L}^+ \hat{b}_1$ on U^+ , similarly $b_{|U^-} = \mathcal{L}^+ \hat{b}$. We have $e^\zeta \hat{\varphi} = \hat{\varphi} + \hat{B} \hat{\varphi} + \hat{b}_1$ and $\mathcal{L}^+(e^\zeta \hat{\varphi})(z) = (\mathcal{L}^+ \hat{\varphi})(z - 1)$. The Laplace transform maps the convolution product to the product of functions and the operator of multiplication by $-\zeta$ to the differentiation $\frac{d}{dz}$, hence its action on each term of $\hat{\varphi} + \hat{B} \hat{\varphi} = \hat{\varphi} + \sum_{k \geq 1} \frac{1}{k!} \hat{b}^{*k} * ((-\zeta)^k \hat{\varphi})$ is clear and (10) allows us to exchange \mathcal{L}^+ and \sum :

$$\mathcal{L}^+(\hat{\varphi} + \hat{B} \hat{\varphi}) = \sum_{k \geq 0} \frac{1}{k!} b^k \left(\frac{d}{dz} \right)^k \mathcal{L}^+ \hat{\varphi} = \mathcal{L}^+ \hat{\varphi} \circ (\text{id} + b).$$

It follows that $\mathcal{L}^+ \hat{\varphi}$ is a solution of (5), and hence that v_*^+ satisfies (4).

Observe that $\mathcal{D}_{R,\delta}^+ \subset U^+$ as soon as there exists δ_0 such that $\delta < \delta_0$ and $R \geq R_0(\delta_0) / \sin(\delta_0 - \delta)$. The 1-Gevrey asymptotic property is standard in the Borel–Laplace summation. It implies $\mathcal{L}^+ \hat{\varphi} = O(z^{-1})$, $\frac{d}{dz}(\mathcal{L}^+ \hat{\varphi}) = O(z^{-2})$, and also the univalence of v_*^+ in $\mathcal{D}_{R,\delta}^+$ for R large enough and the statement on its image, because $|\frac{v_*^+(z)}{z}| \rightarrow 1$ and $\frac{v_*^+(z)}{|v_*^+(z)|} \sim \frac{z}{|z|}$.

For any univalent solution v^+ of (4) on $\mathcal{D}_{R,\delta}^+$, the univalent function $v^+ \circ (v_*^+)^{-1} : \mathcal{D}_{R',\delta'}^+ \rightarrow \mathbb{C}$ conjugates $\text{id} + 1$ with itself, hence is of the form $\text{id} + P$ with a 1-periodic P and extends univalently to \mathbb{C} , hence P is constant. For v_*^- , the proof is similar. \square

Remark 3. The first use of the Borel–Laplace summation for obtaining Fatou coordinates is in [3, §9c]. The asymptotic property without the Gevrey qualification can be found in earlier works by G. Birkhoff, G. Szekeres, T. Kimura, and J. Écalle – see [4]; see [2] for a recent independent proof and an application to numerical computations.

For any pair of Fatou coordinates (v^+, v^-) , we can take $\delta' < \delta$ and large R, R' so that $\mathcal{D}_{R,\delta}^+ \cap (v^-)^{-1}(\mathcal{D}_{R',\delta'}^-)$ has upper and lower connected components \mathcal{Q}^{up} and \mathcal{Q}^{low} . The lifted horn maps are $h^{\text{up}} := v^+ \circ (v^-)^{-1}|_{\mathcal{Q}^{\text{up}}}$, which conjugates $\text{id} + 1$ with itself, hence is of the form $\text{id} + P^{\text{up}}$ with a 1-periodic P^{up} and extends to an upper half-plane, and $h^{\text{low}} := v^+ \circ (v^-)^{-1}|_{\mathcal{Q}^{\text{low}}}$ with similar properties in a lower half-plane. For the normalized pair (v_*^+, v_*^-) , we get $h_*^{\text{up/low}}(Z) = Z + \rho(\text{Log}^+ Z - \text{Log}^- Z) + o(1)$, where $\text{Log}^+ - \text{Log}^- \equiv 0$ in \mathcal{Q}^{up} and $\equiv -2\pi i$ in \mathcal{Q}^{low} , which puts a constraint on the form of the Fourier expansion of $h_*^{\text{up/low}} - \text{id}$:

$$h_*^{\text{up}}(Z) - Z = \sum_{m \geq 1} A_m e^{2\pi i m Z}, \quad h_*^{\text{low}}(Z) - Z = -2\pi i \rho + \sum_{m \geq 1} A_{-m} e^{-2\pi i m Z}. \tag{12}$$

Definition 2. The coefficients $A_m, m \in \mathbb{Z}^*$, are called the Écalle–Voronin invariants of the simple parabolic germ f .

This name is motivated by Écalle–Voronin’s classification result [3,8,5]: (i) two simple parabolic germs f, g are analytically conjugate if and only if there exists $c \in \mathbb{C}$ such that, for all $m, A_m(g) = e^{2\pi i m c} A_m(f)$ (direct consequence of the results of this section); (ii) any pair of Fourier series of the form $(\sum_{m \geq 1} A_m e^{2\pi i m Z}, A_0 + \sum_{m \geq 1} A_{-m} e^{-2\pi i m Z})$, where the first (resp. second) one is holomorphic in an upper (resp. lower) half-plane, can be obtained as $(h_*^{\text{up}} - \text{id}, h_*^{\text{low}} - \text{id})$ for a simple parabolic germ f (this part of the result is more difficult).

By definition, the horn maps $H_*^{\text{up/low}}(w)$ are the lifted horn maps $h_*^{\text{up/low}}$ represented in the coordinate $w = e^{2\pi i Z}$, i.e. $H_*^{\text{up/low}}(w) := \exp(2\pi i h_*^{\text{up/low}}(z))$, so that one gets a germ at 0 and a germ at ∞ on the Riemann sphere:

$$H_*^{\text{up}}(w) = w + 2\pi i A_1 w^2 + O(w^3) \in w\mathbb{C}\{w\}, \quad H_*^{\text{low}}(w) = e^{4\pi^2 \rho} (w + 2\pi i A_{-1} + O(w^{-1})) \in w\mathbb{C}\{w^{-1}\}.$$

Some authors also refer to them as the Écalle–Voronin invariants. The parabolic renormalization operator \mathcal{R} , of great importance in recent developments of complex dynamics, is defined by $\mathcal{R} : F \mapsto H_*^{\text{up}}$ (recall that $F(w) = w + cw^2 + O(w^3) \in w\mathbb{C}\{w\}$ was assumed to be a simple parabolic germ at 0, i.e. $c \neq 0$, and notice that $\mathcal{R}F$ is itself a parabolic germ at 0, but not necessarily simple: this requires $A_1 \neq 0$).

In this article, we have only exploited the information on the principal branch of $\hat{\varphi}$ given by Theorem 1. The reader is referred to [1] for an investigation of the singularities of all the branches of $\hat{\varphi}$ and their relation to the Écalle–Voronin invariants, which shows how the horn maps are encoded in the Borel plane.

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