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The direct image of the relative dualizing sheaf needs not be semiample

L'image directe du faisceau dualisant relatif n'est pas nécessairement semi-ample

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ABSTRACT

We provide details for the proof of Fujita's second theorem and prove that for a Kähler fibre space $f : X \to B$ over a smooth projective curve B, the direct image of the relative dualizing sheaf $V := f_* \omega_{X/B}$ is the direct sum of an ample and a unitary flat bundle. We also show that V needs not be semiample, which is our main result.

RÉSUMÉ

Nous donnons des détails sur la démonstration du second théorème de Fujita et nous montrons que l'image directe du fibré canonique relatif $V := f_* \omega_{X/B}$ d'une fibration $f : X \to B$ sur une courbe B est la somme directe d'un fibré vectoriel ample et d'un fibré vectoriel unitairement plat si l'espace total X est une variété kählérienne compacte. Nous montrons en outre que V n'est en général pas semi-ample, ce qui constitue notre résultat principal.

1. Introduction

An important progress in classification theory was stimulated by a theorem of Fujita, who showed [3] that if X is a compact Kähler manifold and $f : X \to B$ is a fibration onto a smooth projective curve B (i.e., f has connected fibres), then the direct image of the relative dualizing sheaf $V := f_* \omega_{X|B}$ is a numerically semipositive vector bundle on B (over a curve, this is equivalent to saying that the bundle is nef). In this note, which is an abridged version of the article [1], we study further properties of V, related to semipositivity.

Recall that a vector bundle V on a curve is numerically semipositive if and only if every quotient bundle Q of V has degree $\deg(Q) \geqslant 0$, and V is ample if and only if every quotient bundle Q of V has degree $\deg(Q) > 0$ ([9], Theorem 2.4, cf. [1], Prop. 7, see also [15]). In the note [4], Fujita announced the following stronger result (in fact, a flat unitary bundle is numerically positive, cf. [1], Thm. 9):

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Theorem 1.1 *(Fujita's second theorem). Let $f : X \to B$ be a fibration of a compact Kähler manifold X over a projective curve B, and consider the direct image sheaf $V := f_* \omega_{X|B}$. Then V splits as a direct sum $V = A \oplus Q$, where A is an ample vector bundle and Q is a unitary flat bundle.*[1]

Fujita sketched the proof, but referred to a forthcoming article concerning the positivity of the so-called local exponents which however did not appear since. A first purpose of this article is to outline in Section 2 the missing details for the proof of the second theorem of Fujita, which are fully given in [1]. It is important to have in mind Fujita's second theorem in order to understand the question posed by Fujita in 1982 ([10], Problem 5): *Is the direct image $V := f_* \omega_{X|B}$ semi-ample*? In our particular case, where $V = A \oplus Q$ with A ample and Q unitary flat, it simply means that the representation of the fundamental group $\rho : \pi_1(B) \to U(r, \mathbb{C})$ associated with the flat bundle Q has finite image ([1], Thm. 9). The second aim of this article is to outline the proof of [1], Thm. 3, stating that this question has a negative answer:

Theorem 1.2. *There exists a surface X endowed with a fibration $f : X \to B$ onto a curve B of genus $\geqslant 3$, and with fibres of genus 6, such that $V := f_* \omega_{X|B}$ splits as a direct sum $V = A \oplus Q_1 \oplus Q_2$, where the summands Q_i $(i = 1, 2)$ are flat unitary rank-2 bundles having infinite monodromy group and where A is ample. In particular, V is not semi-ample.*

2. Fujita's second theorem

Let B be a smooth complex projective curve. A holomorphic vector bundle over it is identified with its sheaf of holomorphic sections. Assume now that $f : X \to B$ is a fibration of a compact Kähler manifold X over B, and consider the invertible sheaf $\omega := \omega_{X|B} = \mathcal{O}_X(K_X - f^* K_B)$. By Hironaka's theorem, there is a sequence of blow ups with smooth centres $\pi : \hat{X} \to X$ such that $\hat{f} := f \circ \pi : \hat{X} \to B$ has the property that all singular fibres F are such that $F = \sum_i m_i F_i$, and $F_{\mathrm{red}} = \sum_i F_i$ is a normal crossing divisor. Since $\pi_* \mathcal{O}_{\hat{X}}(K_{\hat{X}}) = \mathcal{O}_X(K_X)$, we obtain $\hat{f}_* \omega_{\hat{X}|B} = \hat{f}_* \mathcal{O}_{\hat{X}}(K_{\hat{X}} - \hat{f}^* K_B) = f_* \mathcal{O}_X(K_X - f^* K_B) = f_* \omega_{X|B}$. Therefore, we shall assume that all the reduced fibres of f are normal crossing divisors. By [12], there exists a cyclic Galois covering of B, $B' \to B = B'/G$, such that the normalization X'' of the fibre product $B' \times_B X$ admits a resolution $X' \to X''$ such that the resulting fibration $f' : X' \to B'$ has all the fibres which are reduced and normal crossing divisors. It is proved in [1], Prop. 13, that the sheaf $V' := f'_* \omega_{X'|B'}$ is a subsheaf of the sheaf $u^*(V)$, where $V := f_* \omega_{X|B}$, and the cokernel $u^*(V)/V'$ is concentrated on the set of points corresponding to singular fibres of f'. In particular, since V and V' are semipositive by Fujita's first theorem, *if V' satisfies the property that for each degree 0 quotient bundle Q' of V' then there is a splitting $V' = E' \oplus Q'$ for the projection $p : V' \to Q'$ and Q' is unitary flat, then V' splits as the direct sum $V' = A \oplus Q$, where A is an ample vector bundle and Q is flat unitary bundle, and the same conclusion holds also for V* (cf. [1], Prop. 13).

Theorem 2.1. *(See Fujita, [4].) Let $f : X \to B$ be a fibration of a compact Kähler manifold X over a projective curve B, and consider the direct image sheaf $V := f_* \omega_{X|B}$. Then V splits as a direct sum $V = A \oplus Q$, where A is an ample vector bundle and Q is a unitary flat bundle.*

Proof. By the above discussion it suffices to prove the theorem in the semistable case. Let n be the dimension of X. Let V^* denote the restriction of V to the noncritical locus B^* of f and let $\mathcal{H}^* = (\mathcal{H}^*, \nabla, F)$ denote the variation of polarized Hodge structures underlying the local system $R^{n-1} f_*(\mathbb{C})$ such that $V^* = F^{n-1}(\mathcal{H}^*)$. Let $\mathcal{DH}$ be the canonical extension of $\mathcal{H}^*$ to B, characterized in the semistable case by the nilpotence of the residue matrices of ∇ at the singular points. By the results of Schmid [17], the Hodge filtration extends to a holomorphic filtration of $\mathcal{DH}$, also denoted by F, and it is proved in [11] (cf. also [14]) that $V = F^{n-1}(\mathcal{DH})$. The restriction to V^* of the polarization on $\mathcal{H}^*$ induces the structure of a Hermitian vector bundle on V^*. By [19], Prop. 4.4, for each singular point $s \in S := B \setminus B^*$, there exists a basis of V given by elements σ_j such that their norm in the flat metric outside the punctures grows at most logarithmically (cf. [8]). Hence, for each quotient bundle Q of V, with Q^* denoting the restriction of Q to B^*, the determinant $\det(Q)$ admits a metric h with growth at most logarithmic at the punctures $s \in S$. By [11], Lemma 5, and [16], Prop. 3.4, the degree $\deg(\det(Q))$ of Q is hence given by the integral of the first Chern form $c_1(\det(Q), h) = \Theta_h$ of the singular metric. One has (see [6], Lecture 2):

$$\Theta_{V^*} = \Theta_{\mathcal{H}^*}|_{V^*} + \bar{\sigma}^t \sigma = \bar{\sigma}^t \sigma,$$

with σ denoting the second fundamental form. Griffiths proves ([5], cf. [6], Corollary 5) that the curvature of the dual $(V^*)^\vee$ is semi-negative, since its local expression is of the form $\mathrm{i} h'(z)\, \mathrm{d}\bar{z} \wedge \mathrm{d}z$, where $h'(z)$ is a semipositive definite Hermitian matrix (cf. [1], Section 2, for a discussion on the various notions of curvature positivity). In particular, the curvature Θ_{V^*} of V^* is semipositive. The dual of the principle 'curvature decreases in Hermitian subbundles' [7] implies that the curvature of Q^* is also semipositive. Therefore we can conclude that, since $\deg(Q) = 0$, the quotient Q^* carries a flat connection. Moreover, using the Hermitian splitting, we can view Q^* as a subbundle of V^*. Since the local monodromy of Q^* at the

[1] We remark that, while unitary flatness of a bundle implies numerical semipositivity, flatness alone does not, as shown by the following result ([1], Thm. 4): *Let $f : X \to B$ be a Kodaira fibration, i.e., X is a surface and all the fibres of f are smooth curves not all isomorphic to each other. Then the direct image sheaf $V := f_* \omega_{X|B}$ has strictly positive degree hence $\mathcal{H} := R^1 f_*(\mathbb{C}) \otimes \mathcal{O}_B$ is a flat bundle which is not numerically semipositive.*

singular points $s \in S$ is unipotent (the fibration f being semistable) and moreover unitary, the local monodromy at each $s \in S$ is trivial. Hence we conclude that Q^* has a flat extension to B which we denote by $\hat{Q}$. This extension is tautologically the canonical extension of Q^* and hence we can view $\hat{Q}$ as a subbundle of $\mathcal{DH}$. Since $Q^* \subseteq F^{n-1}(\mathcal{H}^*)$, we have the inclusion $\hat{Q} \subset V = F^{n-1}(\mathcal{DH}) \subset \mathcal{DH}$, and we obtain a homomorphism $\psi : \hat{Q} \to Q$ composing the inclusion $\hat{Q} \to V$ with the surjection $V \to Q$. From the fact that ψ is an isomorphism over B^*, we infer that ψ is an isomorphism: since $\det(\psi)$ is not identically zero, and is a section of a degree zero line bundle. Hence we conclude that the composition of ψ^{-1} with the inclusion $\hat{Q} \to V$ gives then the desired splitting of the surjection $V \to Q$. □

3. A counterexample to Fujita's question

Consider the fibration of projective curves $\varphi : Y \to \mathbb{P}^1_{[x_0,x_1]} =: P$ defined by the minimal resolution of singularities of $\Sigma \to P$, where Σ is the singular μ_7-Galois cover of $\mathbb{P}^1_{[y_0:y_1]} \times P$ (μ_7 denoting the cyclic group of order 7), given by the equation:

$$z_1^7 = y_1 y_0 (y_1 - y_0)(x_0 y_1 - x_1 y_0)^4 x_0^3.$$

Let $P^* = P \setminus \{0, 1, \infty\}$ and let $\tilde{\varphi} : Y^* \to P^*$ denote the restriction of φ to $\varphi^{-1}(P^*) =: Y^*$. The group μ_7 acts fibrewise on the family and $V := \varphi_*(\omega_{Y/P})$ as well as $\mathcal{H}^* = R^1\tilde{\varphi}_*\mathbb{C}_{Y^*} \otimes \mathcal{O}_{P^*}$ splits according to the eigenspaces for the characters $\chi_j : \mu_7 \longrightarrow \mathbb{C}^*$, $\sigma \longmapsto e^{\frac{2\pi i j}{7}}$ $(j = 0, 1, \ldots, 6)$ (we shall denote by V_j, resp. $\mathcal{H}_j^*$, the χ_j-eigensheaf of V, resp. $\mathcal{H}^*$). The fibres $\mathcal{H}_j^*(x)$ of $\mathcal{H}_j^*$ over a point $x \in P^*$ are the vector spaces $H^1(C_x, \mathbb{C})^{\chi_j}$, which have dimension 2, and we have $V_j(x) = H^0(C_x, \Omega^1_{C_x})^{\chi_j} \subseteq \mathcal{H}_j^*(x)$ for $x \in P^*$. It is proven in [1] that in the case $j = 1$ there is a basis of $H^0(C_x, \Omega^1_{C_x})^{\chi_1}$ given by η and $y\eta$, where (in affine coordinates):

$$\eta = y^{-\frac{6}{7}}(y-1)^{-\frac{6}{7}}(x-y)^{-\frac{3}{7}}\mathrm{d}y. \tag{1}$$

This implies that for any $x \in P^*$ there is an equality $V_1(x) = \mathcal{H}_1^*(x)$ which implies an equality of rank-2 vector bundles $\mathcal{H}_1^* = V_1^* := V_1|_{P^*}$ (cf. [2]). The Gauß–Manin connection ∇_1 on $\mathcal{H}_1^* = V_1^*$ (restriction of the Gauß–Manin connection on $\mathcal{H}^*$ to $\mathcal{H}_1^*$) is a flat connection whose local horizontal sections are integrals of the form $g(x) = \int \eta$ $(x \in P^*)$, where η is as in (1). By [13], pp. 163–169, the function $g(x)$ is a solution of the Gauß hypergeometric differential equation $D(\frac{8}{7}, \frac{3}{7}, \frac{9}{7})$ associated with the hypergeometric function ${}_2F_1(\frac{8}{7}, \frac{3}{7}, \frac{9}{7}; x)$. This implies that ∇_1 is isomorphic to the connection associated with $D(\frac{8}{7}, \frac{3}{7}, \frac{9}{7})$. The differential equation $D(\frac{8}{7}, \frac{3}{7}, \frac{9}{7})$ is non-resonant and hence irreducible. Therefore the monodromy group of ∇_1 is irreducible. Moreover, by the Riemann scheme of $D(\frac{8}{7}, \frac{3}{7}, \frac{9}{7})$ (computed as in [13], p. 164) the local monodromy of ∇_1 at the punctures $0, 1 \in P$ is a homology of order 7 and hence is of order 7 in the associated projective linear group. Hence, by the results of Schwarz [18], the monodromy of ∇_1 is infinite. Consider now a ramified covering $\psi : B \to P$, locally at each branch point $0, 1, \infty$ of type $x \mapsto x^7$, and let $\tilde{\psi} : B^* := \psi^{-1}(P^*) \to P^*$ denote the restriction of ψ to $\psi^{-1}(P^*)$. Let $f : X \to B$ be the minimal resolution of the fibre product $B \times_P Y \to B$. Again, the cyclic group μ_7 acts fibrewise on X and it follows fibre-by-fibre that the restriction of the χ_1-eigensheaf $(f_*\omega_{X/B})^{\chi_1}$ to B^* coincides with the pullback of the flat bundle $\tilde{\psi}^*(V_1^*)$. The fibration f has only three singular fibres, but around them the local monodromy of $(f_*\omega_{X/B})^{\chi_1}|_{B^*} = \tilde{\psi}^*(V_1^*)$ is trivial, because the local monodromy of ∇_1 at $0, 1, \infty$ is of order 7. Therefore the vector bundle $(f_*\omega_{X/B})^{\chi_1}|_{B^*}$ extends to a vector bundle $Q_1 \subseteq f_*\omega_{X/B}$ on B carrying a flat connection. But since the monodromy of ∇_1 is infinite, the monodromy of the flat connection on Q_1 is also infinite. Hence Q_1 is a flat (and unitary) summand in $f_*\omega_{X/B}$ with infinite monodromy. The same arguments can be carried out for the character χ_2, leading to another flat summand Q_2 in $f_*\omega_{X/B}$ having also infinite monodromy, and hence leading to the proof of Theorem 1.2.

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