C. R. Acad. Sci. Paris, Ser. I 352 (2014) 179–182

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Lie algebras/Algebraic geometry

Remarks on level-one conformal blocks divisors

Remarques sur les diviseurs associés aux blocs conformes de niveau un

Swarnava Mukhopadhyay

Department of Mathematics, University of Maryland, College Park, MD 20742-4015, USA

ARTICLE INFO

Article history:
Received 24 October 2013
Accepted after revision 9 January 2014
Available online 3 February 2014

Presented by Claire Voisin

ABSTRACT

We show that conformal blocks divisors of type B_r and D_r at level one are effective sums of boundary divisors of $\overline{M}_{0,n}$. We also prove that the conformal blocks divisor of type B_r at level one with weights $(\omega_1,\dots,\omega_1)$ scales linearly with the level.

RÉSUMÉ

Nous montrons que les diviseurs des blocs conformes de type B_r et C_r en niveau un sont des sommes effectives de diviseurs de bord de $\overline{M}_{0,n}$. Nous démontrons également que le diviseur des blocs conformes de type B_r en niveau 1, et avec poids $(\omega_1,\dots,\omega_1)$, croît linéairement avec le niveau.

1. Introduction

Let $\mathfrak{g}$ be a simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra and $P_\ell(\mathfrak{g})$, the set of dominant integral weights of $\mathfrak{g}$ at level ℓ. Consider an n-tuple $\boldsymbol{\Lambda}=(\Lambda_1,\dots,\Lambda_n)$, where $\Lambda_i\in P_\ell(\mathfrak{g})$. Corresponding to this data, consider conformal blocks bundles $\mathbb{V}_{\boldsymbol{\Lambda}}(\mathfrak{g},\ell)$ on $\overline{M}_{0,n}$. The first Chern classes of the bundles $\mathbb{V}_{\boldsymbol{\Lambda}}(\mathfrak{g},\ell)$ are denoted by $\mathbb{D}(\boldsymbol{\Lambda},\mathfrak{g},\ell)$. We refer the reader to C. Sorger [7] for a detailed description of conformal blocks bundles. Thanks to the work of N. Fakhruddin [2], conformal blocks divisors play a central role in the birational geometry of $\overline{M}_{0,n}$.

In this note, we focus on level-one conformal blocks divisors of type B_r and D_r. N. Fakhruddin has shown that conformal blocks divisors at level one are often extremal in the Nef cone of $\overline{M}_{0,n}$. Further, it follows directly from Chern class formulas of Fakhruddin (cf. [2]) that conformal blocks divisors at level one for A_1, A_2, E_6, E_7, E_8, G_2 and F_4 are effective sums of boundary divisors. We refer the reader to Sections 5.2.5–5.2.8 in [2] for more details. It was recently proved in [3] that level-one conformal blocks divisors of type A_r are effective combinations of boundary divisors. We prove the following:

Theorem 1.1. *Conformal blocks divisors of type B_r and D_r at level one are effective combinations of boundary divisors.*

We do not know whether conformal blocks divisors of type C_r at level one are effective combinations of boundary divisors. We believe that this is closely related to the same question about conformal blocks of A_1 at level r due to rank-level duality. Further it will be very interesting to find a non-boundary conformal blocks divisor on $\overline{M}_{0,n}$.

E-mail address: swarnava@umd.edu.

1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2014.01.003

Remark 1. Inspired by Theorem 1.1, it is natural to look for a section of the conformal block bundles of type B_r and D_r at level one which vanishes only on the boundary of $\overline{M}_{0,n}$. Explicit construction of such a section might help in answering the above questions.

We now discuss the behavior of certain conformal blocks for type B_r at level one under scaling.

Theorem 1.2. *Let n be an even integer and $\boldsymbol{\Lambda}$ be the n-tuple of weights $(\omega_1, \dots, \omega_1)$ of B_r at level one. Then we have the following equality in* $\mathrm{Pic}(\overline{M}_{0,n})$:

$$\mathbb{D}(N\boldsymbol{\Lambda}, B_r, N) = N \cdot \mathbb{D}(\boldsymbol{\Lambda}, B_r, 1),$$

where N is a positive integer and $N\boldsymbol{\Lambda}$ is the n-tuple of level-N weights $(N\omega_1, \dots, N\omega_1)$.

Remark 2. It is interesting and challenging to classify the images of morphisms induced by conformal blocks divisors. We hope that the weight functions constructed in the proof of Theorem 1.1 may be used to identify the image of $\overline{M}_{0,n}$ as "Veronese quotients" and shed light on the cone of conformal blocks divisors in type B_r and D_r.

2. Proof of Theorem 1.1

Consider the complete graph $\Gamma[n]$ on n-vertices. Let $\Gamma[S]$ denote the set of edges of $\Gamma[n]$ and assume that the vertices of $\Gamma[n]$ are labeled by the set $[n] = \{1, \dots, n\}$. To every edge s of $\Gamma[n]$, we assign a rational number $w(s)$. We denote by $w(i)$, the total weight of all the edges passing through the vertex i. For a partition I, J of $[n]$ with $|I|, |J| \geqslant 2$, consider the set $V(I|J)$ of all vertices that start at I and end in J. We denote the total weight of the edges in $V(I|J)$ by $w(I|J)$.

Any divisor D on $\overline{M}_{0,n}$ can be written in the form $\sum_{i=1}^{n} a_i\psi_i - \sum_{I,J} c_{I,J} D_{I,J}$, where ψ_i is the i-th ψ class and $D_{I,J}$ is the boundary divisor corresponding to a partition I, J of $[n]$. The following lemma is well known. We refer the reader to [3] for a proof.

Lemma 2.1. *A divisor D is $\mathbb{Q}$-equivalent to an effective combination of boundary divisors on $\overline{M}_{0,n}$ if and only if there exists a $\mathbb{Q}$-valued weight function $w : \Gamma[S] \to \mathbb{Q}$ such that $w(i) = a_i$ and $w(I|J)$ is at least $c_{I,J}$ for all partitions I, J of $[n]$ with $|I|, |J| \geqslant 2$.*

Let $\Lambda \in P_\ell(\mathfrak{g})$. We define the trace anomaly $\Delta_\Lambda(\mathfrak{g}, \ell)$ of Λ to be the following:

$$\Delta_\Lambda(\mathfrak{g}, \ell) := \frac{(\Lambda, \Lambda + 2\rho)}{2(g^* + \ell)},$$

where g^* is the dual Coxeter number of $\mathfrak{g}$, ρ is the half sum of positive roots of $\mathfrak{g}$ and $(\,,)$ is the Cartan Killing form normalized such that $(\theta, \theta) = 2$ for the longest root θ.

The author in [6] rewrote Fakhruddin's formula for $\mathbb{D}(\boldsymbol{\Lambda}, \mathfrak{g}, \ell)$ and expressed it as $\sum_{i=1}^{n} a_i\psi_i - \sum_{I,J} c_{I,J} D_{I,J}$, where:

$$a_i = \Delta_{\Lambda_i}(\mathfrak{g}, \ell) \cdot \mathrm{rk}\,\mathbb{V}_{\boldsymbol{\Lambda}}(\mathfrak{g}, \ell) \quad \text{and} \quad c_{I,J} = \sum_{\Lambda \in P_\ell(\mathfrak{g})} \Delta_\Lambda(\mathfrak{g}, \ell) \cdot \mathrm{rk}\,\mathbb{V}_{\boldsymbol{\Lambda}_I, \Lambda}(\mathfrak{g}, \ell) \cdot \mathrm{rk}\,\mathbb{V}_{\boldsymbol{\Lambda}_J, \Lambda^*}(\mathfrak{g}, \ell),$$

and $\boldsymbol{\Lambda}_I \subset \boldsymbol{\Lambda}$ denotes the set of weights Λ_i such that $i \in I$. To complete the proof of Theorem 1.1, it is enough to construct a $\mathbb{Q}$-valued weight function w on $\Gamma[S]$ satisfying the hypothesis of Lemma 2.1. In the next two sections, we give explicit constructions of the weight functions w for type B_r and D_r, respectively.

3. Weight function for B_r at level one

The level-one weights of B_r are ω_0, ω_1 and ω_r. We ignore ω_0 completely due to "propagation of vacua". The trace anomaly of the level-one weights are $\Delta_{\omega_0} = 0$, $\Delta_{\omega_1} = 1/2$ and $\Delta_{\omega_r} = (2r+1)/16$.

Let n_1, n_2 be the number of ω_1's, ω_r's in $\boldsymbol{\Lambda}$ respectively. If either n_1 or n_2 is zero, then the conformal blocks divisor is symmetric. Hence, by a theorem of [5], it is an effective combination of boundary divisors. If n_2 is odd, then the conformal blocks is zero-dimensional. Hence we assume that $n_1 > 0$ and $n_2 = 2m$ is a positive even number. The dimension of the corresponding conformal blocks at level one is 2^{m-1}.

Remark 3. Let I, J be a partition of $[n]$ and let the number of ω_r's in I be even, then the conformal blocks decompose as the direct sum $\mathbb{V}_{\boldsymbol{\Lambda}_I,\omega_0}(B_r, 1) \otimes \mathbb{V}_{\boldsymbol{\Lambda}_J,\omega_0}(B_r, 1) \oplus \mathbb{V}_{\boldsymbol{\Lambda}_I,\omega_1}(B_r, 1) \otimes \mathbb{V}_{\boldsymbol{\Lambda}_J,\omega_1}(B_r, 1)$ each of dimension 2^{m-2}. On the other hand, if the number of ω_r's is odd, then it is isomorphic to $\mathbb{V}_{\boldsymbol{\Lambda}_I,\omega_r}(B_r, 1) \otimes \mathbb{V}_{\boldsymbol{\Lambda}_J,\omega_r}(B_r, 1)$.

We now describe the weight function $w(s)$ for type B_r associated with the complete graph $\Gamma[n]$, whose vertices are marked by Λ_i.

(i) $w(s) = \frac{\Delta_{\omega_1} 2^{m-1}}{n_1-1} - \frac{2^{m-1}}{n_1(n_1-1)}$, where s is an edge joining two vertices labeled by ω_1.
(ii) $w(s) = \frac{\Delta_{\omega_r} 2^{m-1}}{n_2-1} - \frac{2^{m-1}}{n_2(n_2-1)}$, where s is an edge joining two vertices labeled by ω_r.
(iii) $w(s) = \frac{2^{m-1}}{n_1 n_2}$, otherwise.

It is clear that the flow through every vertex is $2^{m-1}\Delta_{\Lambda_i}(B_r, 1)$. Let I, J be a partition of the set $[n] = \{1, \ldots, n\}$ and suppose a_1, b_1 are the numbers of ω_1's in I and J respectively. Further let $a_2 = |I| - a_1$ and $b_2 = |J| - b_1$. The total flow $w(I|J)$ through the partition I, J is given by the following:

$$\frac{a_1 b_1}{n_1 - 1}\left(\Delta_{\omega_1} 2^{m-1} - \frac{2^{m-1}}{n_1}\right) + \frac{(a_1 b_2 + a_2 b_1) 2^{m-1}}{n_1 n_2} + \frac{a_2 b_2}{n_2 - 1}\left(\Delta_{\omega_r} 2^{m-1} - \frac{2^{m-1}}{n_2}\right). \tag{1}$$

By direct computation, we get the following:

Lemma 3.1.

$$\frac{a_1 b_2 + a_2 b_1}{n_1} - \frac{a_2 b_2}{n_2 - 1} = \frac{a_1 b_2 (b_2 - 1) + b_1 a_2 (a_2 - 1)}{n_1 (n_2 - 1)},$$
$$\frac{a_2 b_2}{n_2 - 1} - 1 = \frac{(a_2 - 1)(b_2 - 1)}{n_2 - 1},$$

where $a_1, a_2, b_1, b_2, n_1, n_2$ are as above.

The following proposition and Remark 3 tell us that the function $w(s)$ satisfies the hypothesis of Lemma 2.1.

Proposition 3.1. *The total flow $w(I|J)$ satisfies the following*:

(i) $w(I|J) \geqslant \Delta_{\omega_r} 2^{m-1}$, *when a_2, b_2 are odd.*
(ii) $w(I|J) \geqslant \Delta_{\omega_1} 2^{m-2}$, *when a_2, b_2 are even.*

Proof. The proof of the proposition is a case by case analysis. We give the details below for completeness.

(i) Let us assume that a_2 and b_2 are both positive odd integers. First we observe that $\Delta_{\omega_1} 2^{m-1} - \frac{2^{m-1}}{n_1} \geqslant 0$, if $n_1 > 1$. By the definition of $w(I|J)$, we get the following:

$$\begin{aligned} & w(I|J) - \Delta_{\omega_r} 2^{m-1} \\ & = \frac{a_1 b_1}{n_1 - 1}\left(\Delta_{\omega_1} 2^{m-1} - \frac{2^{m-1}}{n_1}\right) + \left(\frac{a_1 b_2 + a_2 b_1}{n_1} - \frac{a_2 b_2}{n_2 - 1}\right)\frac{2^{m-1}}{n_2} + \left(\frac{a_2 b_2}{n_2 - 1} - 1\right)\Delta_{\omega_r} 2^{m-1}. \end{aligned}$$

We apply Lemma 3.1 to conclude that $w(I|J) \geqslant 0$. If $n_1 = 1$, then either a_1 or b_1 is zero. The proof in this case follows similarly from Lemma 3.1.

(ii) Let us assume that a_2 and b_2 are both even integers. The proof in this case is further divided into the following two cases.
(a) Let $a_2 = 0$. If b_1 is zero, then the proof follows directly by an easy calculation. Thus we can assume that both a_1 and b_1 are non-zero. We consider the following:

$$w(I|J) - \Delta_{\omega_1} 2^{m-2} = \left(\frac{a_1 b_1}{n_1 - 1} - \frac{1}{2}\right)\Delta_{\omega_1} 2^{m-1} + \frac{a_1}{n_1}\left(1 - \frac{b_1}{n_1 - 1}\right) 2^{m-1}.$$

We apply Lemma 3.1 to conclude that $w(I|J) \geqslant 0$. The case when $b_2 = 0$ is similar.

(b) We are reduced to the case when a_2 and b_2 are both positive even integers. We observe that for $r \geqslant 2$, the trace anomaly $\Delta_{\omega_r} \geqslant 1/4$. We consider the following:

$$\begin{aligned} w(I|J) - \Delta_{\omega_1} 2^{m-2} \geqslant & \frac{a_1 b_1}{n_1 - 1}\left(\Delta_{\omega_1} 2^{m-1} - \frac{2^{m-1}}{n_1}\right) \\ & + \left(\frac{a_1 b_2 + a_2 b_1}{n_1} - \frac{a_2 b_2}{n_2 - 1}\right)\frac{2^{m-1}}{n_2} + \left(\frac{a_2 b_2}{n_2 - 1} - 1\right)\Delta_{\omega_r} 2^{m-1}. \end{aligned}$$

Now the proof follows directly from Lemma 3.1. □

4. Weight function for D_r at level one

The level-one weights of D_r are ω_0, ω_1, ω_{r-1} and ω_r. The trace anomalies of the weights are given as $\Delta_{\omega_1} = 1/2$, $\Delta_{\omega_{r-1}} = \Delta_{\omega_r} = r/8$ and $\Delta_{\omega_0} = 0$. We ignore ω_0 due to "propagation of vacua".

First we observe that conformal blocks divisors of D_3 at level one are up to scaling same as conformal blocks divisors of A_3 at level one. These are all boundary divisors by [3]. Hence we assume that $r > 3$. Let n_1 be the number of ω_1's in $\boldsymbol{\Lambda}$ and $n_2 = n - n_1$. As before, we can assume that $n_1 \neq 0$ and $n_2 > 1$. The case when $n_1 = 0$ follows from an obvious modification of the weight function. It follows from [2] that level-one conformal blocks of type D_r with weights $\boldsymbol{\Lambda}$ are one-dimensional if and only if $\sum_{i=1}^{n} \Lambda_i$ is in the root lattice of D_r and zero otherwise. We now describe the weight function $w(s)$ for type D_r associated with the complete graph $\Gamma[n]$, whose vertices are marked by Λ_i.

(i) $w(s) = \frac{\Delta_{\omega_1}}{n_1-1} - \frac{1}{n_1(n_1-1)}$, where s is an edge joining two vertices labeled by ω_1.
(ii) $w(s) = \frac{\Delta_{\omega_r}}{n_2-1} - \frac{1}{n_2(n_2-1)}$, where s is an edge joining two vertices labeled either by ω_{r-1} or ω_r.
(iii) $w(s) = \frac{1}{n_1 n_2}$, otherwise.

It is clear that the flow through every vertex is Δ_{Λ_i}. Let I, J be a partition of the set $[n] = \{1, \ldots, n\}$ and suppose a_1, b_1 are the numbers of ω_1's in I and J respectively. Further let $a_2 = |I| - a_1$ and $b_2 = |J| - b_1$. The total flow $w(I|J)$ through the partition I, J is given by the following:

$$\frac{a_1 b_1}{n_1 - 1}\left(\Delta_{\omega_1} - \frac{1}{n_1}\right) + \frac{a_2 b_2}{n_2 - 1}\left(\Delta_{\omega_r} - \frac{1}{n_2}\right) + \frac{a_1 b_2 + a_2 b_1}{n_1 n_2}. \tag{2}$$

Since the rank of the bundle $\mathbb{V}_{\boldsymbol{\Lambda}}(D_r, 1)$ is one, it follows that for a partition I, J of $[n]$, there is exactly one $\Lambda \in P_1(\mathfrak{g})$ such that $\mathbb{V}_{\boldsymbol{\Lambda}_I, \Lambda}(D_r, 1)$ and $\mathbb{V}_{\boldsymbol{\Lambda}_J, \Lambda^*}(D_r, 1)$ are both of rank one. It is easy to see that $w(s)$ satisfies the hypothesis of Lemma 2.1 from the following proposition:

Proposition 4.1. *The total flow across a partition satisfies the following properties*:

(i) $w(I|J) \geqslant \Delta_{\omega_1}$, *if a_2 or b_2 is zero.*
(ii) $w(I|J) \geqslant \Delta_{\omega_r}$, *otherwise.*

The proof follows by a direct computation and Lemma 3.1. We skip the details.

5. Proof of Theorem 1.2

Let σ be the non-trivial affine Dynkin diagram automorphism of type B_r. The automorphism σ sends $N\omega_1$ to $N\omega_0$, where ω_0 is the zero-th affine fundamental weight of B_r. Since n is an even integer, it follows from a result of [4] and "propagation of vacua" that the rank of the conformal blocks bundle $\mathbb{V}_{N\boldsymbol{\Lambda}}(B_r, N)$ is one. One the other hand, it is well known that the rank of the conformal blocks bundle $\mathbb{V}_{\boldsymbol{\Lambda}}(B_r, 1)$ is also one if n is even and zero otherwise. We refer the reader to [2] for a proof. Now the proof of Theorem 1.2 follows directly from induction on N and applying Proposition 18.1 in [1].

Acknowledgements

I thank P. Belkale, P. Brosnan and A. Gibney for discussions regarding this note. This work was partially motivated by a question of M. Fedorchuk. After submitting the first version on arXiv, I received an e-mail from N. Fakhruddin with a calculation (unpublished) of his from a couple of years back which also gives the type D result. After the completion of this work, I was informed by M. Fedorchuk about similar results he obtained about conformal blocks being effective combinations of boundary divisors. I thank both of them for their correspondences.

References

[1] P. Belkale, A. Gibney, S. Mukhopadhyay, Quantum cohomology and conformal blocks on $\overline{M}_{0,n}$, arXiv:1308.4906v2.
[2] N. Fakhruddin, Chern classes of conformal blocks, Contemp. Math. 564 (2012) 145–176.
[3] M. Fedorchuk, New Nef divisors on $\overline{M}_{0,n}$, arXiv:1308.5593v1.
[4] J. Fuchs, C. Schweigert, The action of outer automorphisms on bundles of chiral blocks, Commun. Math. Phys. 206 (3) (1999) 691–736.
[5] S. Keel, J. McKernan, Contractible extremal rays on $\overline{M}_{0,n}$, arXiv:alg-geom/9607009v1.
[6] S. Mukhopadhyay, Rank-level duality and conformal block divisors, arXiv:1308.0854v1.
[7] C. Sorger, La formule de Verlinde, in: Séminaire Bourbaki, vol. 1994/1995, Astérisque 237 (1996) 87–114 [Exp. No. 794, 3].