C. R. Acad. Sci. Paris, Ser. I 352 (2014) 229–234

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Harmonic analysis

Uniform bounds of prolate spheroidal wave functions and eigenvalues decay ☆

Bornes uniformes des fonctions d'ondes sphéroidales et décroissance des valeurs propres

Aline Bonami [a], Abderrazek Karoui [b]

[a] *MAPMO–UMR 7349, CNRS–Université d'Orléans, 45067 Orléans, France*
[b] *University of Carthage, Department of Mathematics, Faculty of Sciences of Bizerte, Jarzouna, 7021, Tunisia*

ARTICLE INFO

Article history:
Received 27 July 2013
Accepted after revision 13 January 2014
Available online 5 February 2014

Presented by the Editorial Board

ABSTRACT

The prolate spheroidal wave functions (PSWFs) form a set of special functions with remarkable properties. They are defined on $[-1,1]$ as the bounded eigenfunctions $\psi_{n,c}$ of a Sturm–Liouville differential operator $\mathcal{L}_c$ as well as the eigenfunctions of the linear integral operator $\mathcal{Q}_c$ with kernel $\frac{\sin(c(x-y))}{\pi(x-y)}$. We give new bounds for the values $\psi_{n,c}(0)$, $\psi'_{n,c}(0)$ and $\psi_{n,c}(1)$, which allow us to obtain estimates for the L^p norms of the PSWFs and for eigenvalues of $\mathcal{L}_c$ and $\mathcal{Q}_c$. We get in particular an almost sharp exponential lower decay rate of the eigenvalues of $\mathcal{Q}_c$.

RÉSUMÉ

Les fonctions d'ondes sphéroidales (PSWF) forment un ensemble de fonctions spéciales aux propriétés remarquables. Ces fonctions $\psi_{n,c}$ sont à la fois les fonctions propres d'un opérateur différentiel $\mathcal{L}_c$ de type Sturm–Liouville sur $[-1,1]$ et de l'opérateur intégral $\mathcal{Q}_c$ de noyau $\frac{\sin(c(x-y))}{\pi(x-y)}$. Nous donnons de nouvelles bornes pour les valeurs $\psi_{n,c}(0)$, $\psi'_{n,c}(0)$ et $\psi_{n,c}(1)$, ce qui nous permet d'obtenir des estimations des normes L^p des PSWF ainsi que des valeurs propres des opérateurs $\mathcal{L}_c$ et $\mathcal{Q}_c$. Nous donnons en particulier une borne inférieure presque critique des valeurs propres de l'opérateur $\mathcal{Q}_c$.

1. Introduction

For a given real number $c>0$, called bandwidth, the PSWFs, denoted by $(\psi_{n,c}(\cdot))_{n\geqslant 0}$, are defined as the bounded eigenfunctions of the Sturm–Liouville differential operator $\mathcal{L}_c$, defined on $\mathcal{C}^2([-1,1])$, by:

$$\mathcal{L}_c(\psi)=-\left(1-x^2\right)\frac{d^2\psi}{dx^2}+2x\frac{d\psi}{dx}+c^2x^2\psi. \tag{1}$$

☆ This work was supported in part by the ANR grant "AHPI" ANR-07-BLAN-0247-01, the French–Tunisian CMCU 10G 1503 and the DGRST Research Grant 05 UR 15-02.

E-mail addresses: aline.bonami@univ-orleans.fr (A. Bonami), abderrazek.karoui@fsb.rnu.tn (A. Karoui).

1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2014.01.004

They are also the eigenfunctions of the finite Fourier transform $\mathcal{F}_c$, as well as the ones of the operator $\mathcal{Q}_c = \frac{c}{2\pi}\mathcal{F}_c^*\mathcal{F}_c$, which are defined on $L^2([-1,1])$ by:

$$\mathcal{F}_c(f)(x) = \int_{-1}^{1} e^{icxy} f(y)\,\mathrm{d}y, \qquad \mathcal{Q}_c(f)(x) = \int_{-1}^{1} \frac{\sin(c(x-y))}{\pi(x-y)} f(y)\,\mathrm{d}y. \tag{2}$$

They are normalized so that their $L^2([-1,1])$ norm is equal to 1. We call $(\chi_n(c))_{n\geqslant 0}$ the eigenvalues of $\mathcal{L}_c$, $\lambda_n(c)$ the eigenvalues of $\mathcal{F}_c$ and $\mu_n(c)$ the ones of $\mathcal{Q}_c$. The crucial commutation property of $\mathcal{L}_c$ and Q_c was first observed by D. Slepian, [8], whose name is closely associated with all properties of PSWFs and their associated spectrum. Among their basic properties, we cite their analytic extension to the whole real line and their unique properties to form an orthonormal basis of $L^2([-1,1])$ and an orthonormal basis of B_c, the Paley–Wiener space of c-band-limited functions, defined as L^2 functions whose Fourier transform vanishes outside the interval $[-c,c]$. As a result of these properties, the PSWFs have been used in spectral approximation schemes as well as in various applications from physics and signal processing, see for instance [2,3,6,10].

The asymptotic behavior of the PSWFs is well known for fixed c. Our purpose here is to give uniform bounds, which can be used to find explicit bounds for the decay rate of the eigenvalues $\mu_n(c)$ or for uniform approximation of the PSWFs. See also [4] for an application of uniform bounds in the context of compressed sensing.

We first give uniform bounds for $|\psi_{n,c}(0)|$ or $|\psi'_{n,c}(0)|$, depending on the parity of n. In particular, we find that $|\psi_{n,c}(0)| \leqslant 1$ and $|\psi'_{n,c}(0)| \leqslant \sqrt{\chi_n(c)}$ as long as the quantity $q = c^2/\chi_n(c)$ is smaller than 2 (Theorem 2.1). In Section 3, we also give bounds below for these quantities when $q < 1$. We prove that values at 0 dominate in some way other values of $\psi_{n,c}$: this is the fundamental inequality (9). It leads us to uniform bounds for the L^∞ norm of $\psi_{n,c}$. Other L^p norms are also considered under the condition that:

$$q = c^2/\chi_n(c) \leqslant 1. \tag{3}$$

The oscillatory properties of $\psi_{n,c}$ depend on this condition, which is everywhere present in their study. Osipov has proved in [7] that this is satisfied when $n \geqslant \frac{2c}{\pi}$, while it is not satisfied when $n+1 \leqslant \frac{2c}{\pi}$. Under this condition, the supremum of $|\psi_{n,c}|$ is attained at 1.

An emblematic estimate frequently cited (but not proved) in the literature is given by $|\psi_{n,c}(1)| \leqslant \sqrt{n+\frac{1}{2}}$. This has been numerically justified in [12], while an analytic proof of this inequality seems to be difficult to obtain. We will not prove it either, but the following estimate, valid for $q < 1$,

$$\left|\psi_{n,c}(1)\right| \leqslant \kappa\sqrt{n+1}, \tag{4}$$

with $\kappa = 2.35$, follows from our study.

We finally use our estimates to give bounds below for the eigenvalues $\chi_n(c)$ and $\mu_n(c)$. This follows from the fact that derivatives in c depend on the quantities that we have studied before.

We will write $\kappa_1, \kappa_2, \ldots$ for explicit constants that are used in uniform bounds. We assume, without loss of generality, that $\psi_{n,c}(1) > 0$.

2. The value at zero

Let us recall that $\psi_{n,c}$ is even or odd, depending on the fact that n is even or odd. We may not systematically write the index c for simplification. We recall [7] that, for $q \leqslant 1$, the maximum of $|\psi_n|$ inside $[0,1]$ is attained at 1, while for $q > 1$, its maximum is attained at some point $0 \leqslant y_n < 1/\sqrt{q}$ and $\psi_n(x)$ is strictly positive for $x \in]y_n, 1]$. The first result of this section is the following.

Theorem 2.1. *For all $q \geqslant 0$ we have the bounds*:

$$\left|\psi_n(0)\right|^2 + \chi_n^{-1}\left|\psi_n'(0)\right|^2 \leqslant \begin{cases} 1 & \text{if } 0 \leqslant q \leqslant 2 \\ \frac{q+1}{\sqrt{q}} & \text{if } q > 2. \end{cases} \tag{5}$$

Proof. We use the notation:

$$A = A_{n,c} = \left(\left|\psi_n(0)\right|^2 + \chi_n^{-1}\left|\psi_n'(0)\right|^2\right)^{\frac{1}{2}}. \tag{6}$$

We consider the auxiliary function $z_n(t) = (1-t^2)(1-qt^2)\psi_n^2(t) + \frac{(1-t^2)^2}{\chi_n}(\psi_n'(t))^2$. Using the equation we find that $z_n'(t) = -2t(q+1-2qt^2)(\psi_n(t))^2$, so that:

$$A^2 = 2\int_0^1 s\big(q+1-2qs^2\big)\psi_n^2(s)\,\mathrm{d}s \leqslant \max_{s\in[0,1]}\big(s\big(q+1-2qs^2\big)\big). \tag{7}$$

We conclude easily for (5) when $q<2$. Remark that the constant 1 on the right-hand side of (5) can be replaced by the smaller constant $\max(1-q,4/5)$ for $q<1$.

Let us now assume that $q>2$. The bound $\frac{q+1}{\sqrt{q}}$ in (5) follows from the inequality $A^2 \leqslant 2(q+1)\int_0^1 s\psi_n^2(s)\,\mathrm{d}s$ and the use of the Cauchy–Schwarz inequality together with the fact that $\int_{-1}^1 s^2\psi_n^2(s)\,\mathrm{d}s$ is bounded by q^{-1}. This last bound is an immediate consequence of the identity:

$$\int_{-1}^1 \big(1-s^2\big)\big(\psi_n'(s)\big)^2\,\mathrm{d}s + c^2\int_{-1}^1 s^2\psi_n^2(s)\,\mathrm{d}s = \chi_n. \qquad \square \tag{8}$$

The estimate (5) is not sharp: for Legendre polynomials, that is, when $q=0$, one finds the bound $2/\pi$ for n even, $3/4$ for n odd.

The next proposition will allow us to control many quantities in terms of the values at 0, extending those given in Szegö's book [9], on p. 164, for Legendre polynomials.

Proposition 2.1. *We have the inequality*:

$$\sqrt{\big(1-t^2\big)\big(1-qt^2\big)}\big|\psi_{n,c}(t)\big|^2 \leqslant \big|\psi_{n,c}(0)\big|^2 + \chi_n^{-1}\big|\psi_{n,c}'(0)\big|^2, \quad t\leqslant \min\big(1,\sqrt{1/q}\big). \tag{9}$$

Proof. We set:

$$S(t) = \int_t^{\min(1,\sqrt{1/q})} \sqrt{\frac{1-qx^2}{1-x^2}}\,\mathrm{d}x. \tag{10}$$

By using the substitution $\psi_n(t) = (1-t^2)^{-1/4}(1-qt^2)^{-1/4}U(S(t))$, it can be checked that the equation satisfied by U is given on $(0,S(0)]$ by:

$$U'' + (\chi_n+\theta)U = 0,$$

with $\theta\circ S$ an explicit rational function which is increasing on $[0,\min(1,\frac{1}{\sqrt{q}})]$, see [1]. The decay of S implies that θ is also decreasing, so that $Z_n(s) = |U(s)|^2 + \frac{1}{\chi_n+\theta(s)}|U'(s)|^2$ is increasing. Therefore, $|U(S(t))|^2 \leqslant Z_n(S(t)) \leqslant Z_n(S(0))$ and since $U'(S(0)) = \psi_n'(0)$, then (9) follows at once. $\square$

3. Norms in Lebesgue spaces

From now on, we write $Q_q(t) := (1-t^2)(1-qt^2)$. Norms are taken on $(-1,+1)$. Let us first consider the L^∞ norm.

Theorem 3.1. *For the constant $\kappa_1 = \frac{5^{5/4}}{4} \approx 1.87$, we have the inequality, valid for all $c\geqslant 0$ and $n\geqslant 0$,*

$$\|\psi_{n,c}\|_\infty \leqslant \kappa_1\chi_n(c)^{\frac14}\big(\big|\psi_{n,c}(0)\big|^2 + \chi_n(c)^{-1}\big|\psi_{n,c}'(0)\big|^2\big)^{\frac12}. \tag{11}$$

In particular, if $q\leqslant 1$, then

$$\big|\psi_{n,c}(1)\big| \leqslant \kappa_1\chi_n(c)^{\frac14} \leqslant \kappa_1\sqrt{\frac{\pi}{2}}\sqrt{n+1} \approx 2.35\sqrt{n+1}. \tag{12}$$

Proof. Assume first that $q\leqslant 1$. The maximum of ψ_n is then attained at 1. We use (9) for x_0, a value of $x<1$ sufficiently close to 1, so that $\psi_{n,c}(x_0)$ has the same order as $\psi_{n,c}(1)$. Since $((1-t^2)\psi_n'(t))' = -\chi_n(1-qt^2)\psi_n(t)$, we have:

$$\big|\psi_n'(x)\big| \leqslant \frac{\chi_n}{1-x^2}\int_x^1 \big(1-qt^2\big)\big|\psi_n(t)\big|\,\mathrm{d}t \leqslant \chi_n\big(1-qx^2\big)\psi_n(1). \tag{13}$$

Hence, $\psi_n(1)-\psi_n(x)\leqslant\chi_n\psi_n(1)Q_q(x)$. Moreover, if we choose an $x_0\in[0,1)$ so that $Q_q(x_0)=a\chi_n^{-1}$ for some positive constant a, then from (9), we have:

$$\psi_n(1)\leqslant A\frac{a^{-1/4}\chi_n^{1/4}}{1-a}\leqslant\kappa_1\chi_n^{1/4}.$$

The value $\kappa_1=(5/4)\times 5^{\frac{1}{4}}$ is obtained at the maximum of the quantity $a^{-1/4}(1-a)^{-1}$, that is, $a=1/5$. Moreover, by using the fact that $A\leqslant 1$ and $\chi_n(c)\leqslant(\frac{\pi}{2}(n+1))^2$, see [7], one gets the second inequality given in (12).

Let us now assume that $q>1$. The maximum is attained at $y<\frac{1}{\sqrt{q}}$. If $Q_q(y)>\chi_n^{-1}$, then we conclude directly from (9). So we can assume that $(1-qy^2)^2\leqslant Q_q(y)\leqslant\chi_n^{-1}$, which implies that $y>\frac{1}{\sqrt{q}}(1-\chi_n^{-\frac{1}{2}})$. We proceed as before, by choosing some point $y_0<y$, for which we now have the inequality:

$$0\leqslant\psi_n(y)-\psi_n(y_0)\leqslant\chi_n Q_{q,y}(y_0)\psi_n(y),\qquad Q_{q,y}(x)=(y-x)\big(1-qx^2\big)\leqslant Q_q(x).$$

Next, we choose y_0 so that $Q_{q,y}(y_0)<\chi_n^{-1}/5$, which is possible if $Q_{q,y}(0)=y>\chi_n^{-1}/5$. To prove this inequality for $n\geqslant 1$, it is sufficient to prove that $\chi_n\geqslant\sqrt{q}$. This is straightforward for $q<2$ and follows easily for $q>2$ from the inequality:

$$\frac{\pi n}{2}\leqslant\sqrt{\frac{\chi_n(c)}{q}},\quad q>1 \tag{14}$$

which is due to Osipov (see Proposition 3 in [7] and [1] for a further discussion). From this point, the proof is the same as for $q<1$. □

The constant κ_1 computed here is certainly not optimal, but it is not far from being optimal. For Legendre polynomials, the best constant is asymptotically equal to $\sqrt{\pi/2}$.

Remark 3.1. By combining (5) and (11), one gets:

$$\sup_{x\in[-1,1]}\big|\psi_n(x)\big|\leqslant\kappa_1\begin{cases}\chi_n^{\frac{1}{4}} & \text{if } 0\leqslant q\leqslant 2\\ (c+\chi_n^{\frac{1}{2}})^{\frac{1}{2}} & \text{if } q>2.\end{cases} \tag{15}$$

Let us now consider L^p-norms when $q=c^2/\chi_n<1$. It is well known that the condition $p<4$ ensures uniform estimates in n for Legendre polynomials. The extension of this property to the PSWFs is given by the following theorem.

Theorem 3.2. *For $q<1$ and $p<4$, we have the following bounds*:

$$\|\psi_{n,c}\|_p\leqslant\big\|Q_q^{-1/4}\big\|_p\big(\big|\psi_{n,c}(0)\big|^2+\chi_n(c)^{-1}\big|\psi'_{n,c}(0)\big|^2\big)^{\frac{1}{2}}<\big\|Q_q^{-1/4}\big\|_p. \tag{16}$$

Moreover, for $q<1$, we have:

$$\|\psi_{n,c}\|_4^4\leqslant 2(1-q)^{-1}\big((1-q)\kappa_1^4+\ln\chi_n(c)\big). \tag{17}$$

Proof. The first inequality is a straightforward consequence of (9). For the inequality (17), we cut the integral into two parts, from 0 to $1-\frac{1}{\chi_n}$, then from $1-\frac{1}{\chi_n}$ to 1. For the first integral, we use (9) again, while for the second one we replace $|\psi_n|$ by its upper bound $\psi_n(1)$. □

For $q<1$ we get a lower bound for the value at 0 from (16) written for $p=2$. Let us denote by $\mathbb{K}$ the Legendre complete integral of the first kind $\mathbb{K}(r)=\int_0^1\frac{dt}{\sqrt{1-t^2}\sqrt{1-r^2t^2}}$, $0\leqslant r<1$. Then:

$$\big|\psi_{n,c}(0)\big|^2+\chi_n(c)^{-1}\big|\psi'_{n,c}(0)\big|^2\geqslant\big(2\mathbb{K}(\sqrt{q})\big)^{-1}\geqslant\frac{1}{\ln(\frac{1+q}{1-q})+\pi}. \tag{18}$$

Indeed, the last inequality comes from:

$$\mathbb{K}(r)\leqslant\frac{1}{2}\ln\frac{1+r^2}{1-r^2}+\frac{\pi}{2}, \tag{19}$$

which can be obtained by considering the difference between $\mathbb{K}(r)$ and the integral of $r/(1-r^2t^2)$.

A second bound below for A can be obtained by writing $\frac{1}{2}=\int_0^1|\psi_n(t)|^2\,\mathrm{d}t=\int_0^{x_1}+\int_{x_1}^1$ and choosing x_1 such that $(1-x_1)\kappa_1^2\sqrt{\chi_n}=\frac{1}{4}$. Then:

$$\frac{1}{4}\leqslant\int_0^{x_1}\left|\psi_n(t)\right|^2\mathrm{d}t\leqslant A^2\int_0^{x_1}\left|Q_q(t)\right|^{-\frac{1}{2}}\mathrm{d}t\leqslant\frac{A^2}{4}\left(\ln(\chi_n)+\kappa_2\right),$$

where $\kappa_2=4\ln(4\kappa_1)\approx 8.05$. This leads to the bound below:

$$\left|\psi_{n,c}(0)\right|^2+\chi_n(c)^{-1}\left|\psi_{n,c}'(0)\right|^2\geqslant\left(\ln(\chi_n)+\kappa_2\right)^{-1}. \tag{20}$$

This is better than (18) for χ_n close from c^2 and not too large. We have just proved the following proposition.

Proposition 3.1. *For $q<1$, we have the bound below*:

$$\left|\psi_{n,c}(0)\right|^2+\chi_n(c)^{-1}\left|\psi_{n,c}'(0)\right|^2\geqslant\max\left(\left(\ln\left(\frac{1+q}{1-q}\right)+\pi\right)^{-1},\left(\ln\left(\chi_n(c)\right)+\kappa_2\right)^{-1}\right). \tag{21}$$

We conclude this section by proving a converse inequality. Namely,

Theorem 3.3. *There exists a constant κ_3 such that, when $q<1$,*

$$\left(\left|\psi_{n,c}(0)\right|^2+\chi_n(c)^{-1}\left|\psi_{n,c}'(0)\right|^2\right)^{\frac{1}{2}}\leqslant\kappa_3\|\psi_{n,c}\|_1. \tag{22}$$

We can take $\kappa_3=14$.

Proof. Since $1=\|\psi_n\|_2^2\leqslant\kappa_1\chi_n(c)^{\frac{1}{4}}\|\psi_n\|_1$, there is nothing to prove for small values of n. We assume now that $n\geqslant n_0$, where the integer n_0 is to be fixed later on. We will sketch the proof of a stronger result, namely:

$$A\leqslant\kappa_3\int_{-\frac{1}{2}}^{\frac{1}{2}}\left|\psi_{n,c}(t)\right|\mathrm{d}t.$$

It is well known that the values of the local maxima of $|\psi_n|$ are bounded below by A (see [7]). We also know that the distance of two consecutive zeros of ψ_n is at most $\frac{\pi}{\sqrt{\chi_n+1}}$ (see [7], Theorem 14). So the number of zeros in $(-\frac{1}{2},+\frac{1}{2})$ is at least $k=[\frac{\sqrt{\chi_n+1}}{\pi}]-1$, and the number of local maxima of $\psi_{n,c}$ in the same interval is at least $k-1>0$. We are done if we can prove that, for y such a value where there is a local maximum, $|\psi_n|$ is larger than $A/3$ in an interval having y as one extremity and size at least $\frac{1}{\sqrt{\chi_n}}$, all such intervals being disjoint. We assume that $y>0$ and fix $y'<y$ the largest zero of ψ_n, which is smaller than y. We proceed as in the proof of (13) to get $|\psi_n'(x)|\leqslant\frac{4}{3}|\psi_n(y)|\chi_n(y-x)$ for $x\in(y',y)$, then $|\psi_n(y)-\psi_n(x)|\leqslant\frac{2}{3}|\psi_n(y)|\chi_n(y-x)^2$. On the interval $I=(y-\frac{1}{\sqrt{\chi_n}},y)$, one has $|\psi_n(x)|>\frac{A}{3}$, which we wanted. Moreover I is contained in (y',y), which guarantees the disjointness. The same argument holds when y is negative or zero. Finally, if $y_i, i=1,\dots,k-1$ denote the different local maxima of $|\psi_n|$ in $(-\frac{1}{2},+\frac{1}{2})$, and $I_i=(y_i-\frac{1}{\sqrt{\chi_n}},y_i)$ the corresponding intervals, then we have:

$$\int_{-1/2}^{1/2}\left|\psi_n(t)\right|\mathrm{d}t\geqslant\sum_{i=1}^{k-1}\int_{I_i}\left|\psi_n(t)\right|\mathrm{d}t\geqslant\left(\left[\pi^{-1}\sqrt{\chi_n}\right]-2\right)\frac{A}{3\sqrt{\chi_n}}\geqslant\frac{A}{3\pi}\left(1-\frac{3\pi}{n}\right).$$

We may take for κ_3 any value that dominates $\max(\kappa_1\chi_{n-1}^{1/4},3\pi(1-\frac{3\pi}{n})^{-1})$ for some $n\geqslant n_0$. We find the constant 14 by taking $n=n_0=30$ and using again the bound $\chi_n\leqslant\frac{\pi^2(n+1)^2}{4}$ for the first quantity. $\square$

4. Eigenvalues

In the following proposition, we give a new lower bound of the eigenvalues $\chi_n(c)=\chi_n$ of the differential operator $\mathcal{L}_c$.

Proposition 4.1. *For c and n such that (3) is satisfied, we have*:

$$n(n+1)+(3-2\sqrt{2})c^2\leqslant\chi_n(c)\leqslant n(n+1)+c^2. \tag{23}$$

Proof. The upper bound of (23) is well known [12]. It follows from the fact that $\partial_c\chi_n(c)=2c\int_{-1}^1 t^2|\psi_{n,c}(t)|^2\,\mathrm{d}t$ and $\chi_n(0)=n(n+1)$. Let us prove that the quantity $B=\int_{-1}^1 t^2\psi_{n,c}(t)^2\,\mathrm{d}t$ is bounded below by $3-2\sqrt{2}$, from which we conclude also for the lower bound. If A is defined as above, coming back to the methods of the first section, we remark that, for $q<1$, the auxiliary function $-(1-qt^2)^{-1}z_n$ has its derivative bounded by $2t|\psi_n|^2$, so that one has the inequality:

$$A^2 \leqslant 2\int_0^1 s\psi_n^2(s)\,ds \leqslant B^{\frac{1}{2}}. \tag{24}$$

On the other hand, the use of (9) allows us to conclude that $1-B \leqslant 2A^2$. Hence $B^{\frac{1}{2}}$ is bounded below by the largest solution of the equation $X^2+2X-1=0$, which allows us to conclude. □

We finish by an estimate on the decay rate of the eigenvalues of $\mathcal{F}_c$ and $\mathcal{Q}_c$. We recall that they are related by the relation $\mu_n(c)=\frac{c}{2\pi}|\lambda_n(c)|^2$. It is well known (see [12]) that:

$$\partial_c \ln \mu_n(c) = \frac{2|\psi_{n,c}(1)|^2}{c}.$$

This means that all lower bounds or upper bounds on $|\psi_n(1)|^2$ will give upper bounds or lower bounds on $\mu_n(c)$. It has been proved by Landau [5] that these eigenvalues decay very slowly, for fixed c, as long as $\chi_n(c)<c^2$. More precisely, we have:

$$\mu_n(c) \geqslant 0.4 \quad \text{for } n \leqslant \left[\frac{2c}{\pi}\right]-1; \qquad \mu_n(c) \leqslant 0.6 \quad \text{for } n \geqslant \left[\frac{2c}{\pi}\right]+1. \tag{25}$$

So, in particular, for $c \leqslant \frac{\pi(n+1)}{2}$, we have:

$$\mu_n(c) > 0.4\exp\left(-2\int_c^{\frac{\pi(n+1)}{2}} \left|\psi_{n,\tau}(1)\right|^2 \frac{d\tau}{\tau}\right).$$

When $\tau \leqslant \frac{\pi n}{2}$, we know that $q<1$. For $\frac{\pi n}{2} < \tau < \frac{\pi(n+1)}{2}$, we have $\chi_n(\tau) \geqslant \chi_n(\frac{\pi n}{2}) \geqslant \frac{\pi^2 n^2}{4}$. So $\frac{\tau^2}{\chi_n(\tau)} \leqslant 2$ as soon as $\frac{n+1}{n} \leqslant \sqrt{2}$. We now use (15). Since $\chi_n(\tau)$ is increasing with τ and since $\chi_n(\frac{\pi(n+1)}{2}) \leqslant (\frac{\pi(n+1)}{2})^2$ (recall [7] that for this value $q>1$), we conclude as follows.

Proposition 4.2. *For $n \geqslant 3$ and $c \leqslant \frac{\pi(n+1)}{2}$, we have the inequality*:

$$\mu_n(c) > 0.4\left(\frac{2c}{\pi(n+1)}\right)^{\pi\kappa_1^2(n+1)}. \tag{26}$$

Let us remark that the inequality $\psi_{n,c}(1) \leqslant \sqrt{n+\frac{1}{2}}$, if it was correct, would improve all constants. Precise estimates have been given in [7] by different methods. The asymptotic behavior is well known, see for instance [11].

References

[1] A. Bonami, A. Karoui, Uniform estimates of the prolate spheroidal wave functions, preprint, 2013.
[2] J.P. Boyd, Prolate spheroidal wave functions as an alternative to Chebyshev and Legendre polynomials for spectral element and pseudo-spectral algorithms, J. Comput. Phys. 199 (2004) 688–716.
[3] Q. Chen, D. Gottlieb, J.S. Hesthaven, Spectral methods based on prolate spheroidal wave functions for hyperbolic PDEs, SIAM J. Numer. Anal. 43 (5) (2005) 1912–1933.
[4] L. Gosse, Compressed sensing with preconditioning for sparse recovery with subsampled matrices of Slepian prolate functions, Ann. Univ. Ferrara, Sez. VII: Sci. Mat. 59 (2013) 81–116.
[5] H.J. Landau, H.O. Pollak, Prolate spheroidal wave functions, Fourier analysis and uncertainty—III. The dimension of space of essentially time- and band-limited signals, Bell Syst. Tech. J. 41 (1962) 1295–1336.
[6] L.W. Li, X.K. Kang, M.S. Leong, Spheroidal Wave Functions in Electromagnetic Theory, Wiley–Interscience, 2001.
[7] A. Osipov, Certain inequalities involving prolate spheroidal wave functions and associated quantities, Appl. Comput. Harmon. Anal. 35 (2013) 359–393.
[8] D. Slepian, H.O. Pollak, Prolate spheroidal wave functions, Fourier analysis and uncertainty. I, Bell Syst. Tech. J. 40 (1961) 43–64.
[9] G. Szegö, Orthogonal Polynomials, Colloquium Publications, vol. XXIII, American Mathematical Society, Providence, RI, USA, 1975.
[10] L.L. Wang, Analysis of spectral approximations using prolate spheroidal wave functions, Math. Comput. 79 (2010) 807–827.
[11] H. Widom, Asymptotic behavior of the eigenvalues of certain integral equations. II, Arch. Ration. Mech. Anal. 17 (1964) 215–229.
[12] H. Xiao, V. Rokhlin, N. Yarvin, Prolate spheroidal wave functions, quadrature and interpolation, Inverse Probl. 17 (2001) 805–838.