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A note on homotopy classes of nonsingular vector fields on S^3

Une note sur les classes d'homotopie de champs de vecteurs sans singularité sur S^3

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ABSTRACT

The homotopy class (up to homeomorphism) of nonsingular vector fields on S^3 are in one-to-one correspondence with $\mathbb{N}$ via the homotopy number. We prove that each homotopy class with a nonzero homotopy number can be represented by two nonsingular Morse–Smale vector fields with three periodic orbits. Notice that it is already known that the nonsingular Morse–Smale vector field with two periodic orbits has homotopy number 0.

RÉSUMÉ

Les classes d'homotopie (à homéomorphisme près) de champs de vecteurs sans singularité sur la sphère S^3 sont indexées, via le nombre d'homotopie, par les entiers positifs. Nous montrons que chaque classe de nombre d'homotopie non nul peut être représentée par deux champs de vecteurs de type Morse–Smale sans singularité, avec trois orbites périodiques. Ce résultat est optimal, puisqu'on sait déjà que tout champ avec deux orbites périodiques a 0 pour nombre d'homotopie.

1. Introduction

The celebrated Nielsen–Thurston theorem tells us that one can always find a representor in each isotopy class of homeomorphisms on a given surface such that the representor is (in some sense) the simplest. For a 3-manifold, one can similarly expect to find a "simplest" representor in each homotopy class of nonsingular vector fields on the 3-manifold. But, as remarked by MacKay [4], the notion of "simplest" is still unclear. In particular, the famous counterexample of the Seifert conjecture by Kuperberg [3] implies that each nonsingular vector field on a three manifold is homotopic to a C^∞ vector field without periodic orbit. But, actually the dynamics of the vector fields of Kuperberg are quite complicated, see, for instance, [2]. Therefore, it is still interesting to know whether each nonsingular vector field can be homotopic to a well-known class of vector fields (e.g., Morse–Smale, volume-preserving, pseudo-Anosov vector fields).

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In this paper, we restrict our discussion to nonsingular Morse–Smale vector fields (abbreviated as NMS vector fields). Actually, Yano [7] gave a computable condition to decide for a given homotopy class on a 3-manifold whether it can be homotopic to a NMS vector field. In particular, when we consider S^3, each nonsingular vector field is homotopic to a NMS vector field. Notice that the homotopy class of nonsingular vector fields of S^3 is in one-to-one correspondence with $\pi_3(S^2) \cong \mathbb{Z}$. Furthermore, the homotopy class up to homeomorphism on S^3 is in one-to-one correspondence with $\mathbb{N}$.

Suppose that X is a nonsingular vector field on S^3; we will recall an algorithm used by Dufraine [1] in the next section to compute (or define) the corresponding natural number in $\mathbb{N}$. Such a number of X is denoted by $\mathcal{I}(X)$ and called the *homotopy number* of X in the sense that the homotopy number gives a well-defined isomorphism between the homotopy classes (up to homeomorphism) of nonsingular vector fields on S^3 and $\mathbb{N}$ (more information about the homotopy number can be found in [1] and [5]).

Definition 1.1. Let X be a NMS vector field on S^3. We denote by $\mathcal{NP}(X)$ the number of the periodic orbits of X. For a given $n \in \mathbb{N}$, $\mathcal{M}(n)$ is defined to be the minimal number of $\mathcal{NP}(X)$ for every NMS vector field X on S^3 with $\mathcal{I}(X) = n$.

Now the following question is natural.

Question 1.2.

1. For a given $n \in \mathbb{N}$, what is $\mathcal{M}(n)$?
2. For a given $n \in \mathbb{N}$, how to list all NMS vector fields $\{X\}$ up to topological equivalence such that $\mathcal{I}(X) = n$ and $\mathcal{NP}(X) = \mathcal{M}(n)$?

A complete answer of Question 1.2 means that we can use NMS vector fields to represent nonsingular vector fields on S^3 in the following sense. For a given $n \in \mathbb{N}$, the NMS vector fields $\{X\}$ with $\mathcal{I}(X) = n$ and $\mathcal{NP}(X) = \mathcal{M}(n)$ can be used to represent the homotopy class of nonsingular vector fields with homotopy number n. Such kind of representors make the following senses.

1. Each homotopy class of nonsingular vector fields on S^3 admits such a representor.
2. Their periodic orbits number are lowest in all NMS vector fields with homotopy number n.
3. Each X has quite simple dynamics, since X is a NMS vector field.

For every $n \in \mathbb{N}$, Wilson [6] has constructed a NMS vector field X_n on S^3 such that $\mathcal{I}(X_n) = n$ and $\mathcal{NP}(X_n) \leqslant 6$. Moreover, Dufraine [1] has computed that for every $n \in \mathbb{N}$, there exists a NMS vector field X^n such that $\mathcal{I}(X^n) = n$ and

$$\mathcal{NP}\left(X^n\right) = \begin{cases} 2, & \text{if } n = 0, \\ 3, & \text{if } n = 1, \\ 4, & \text{if } n \geqslant 2. \end{cases}$$

In the last section of [8], the author has given a list (which also can be found in the next section) to completely classify NMS vector fields with 3 periodic orbits on S^3. Here we will use the list to compute their homotopy numbers one-by-one via the algorithm used by Dufraine [1]. We collect these results as the following theorem.

Theorem 1.3. *$\mathcal{M}(n) = 3$ if $n \in \mathbb{N}^+$. On the other hand, for a given $n \in \mathbb{N}^+$, we can list NMS vector fields with 3 periodic orbits and $\mathcal{I}(X) = n$ as follows.*

1. *If $n = 1$, there are two NMS vector fields: $p = 0$ and $p = 1$ in case 2 of the list in Section 2.1.*
2. *If $n = 2k + 1$ $(k \geqslant 1)$, there are two NMS vector fields: $p = -k$ in case 2 and case 4 of the list.*
3. *If $n = 2k$ $(k \geqslant 1)$, there are two NMS vector fields: $p = k + 1$ in case 2 and case 4 of the list.*

Moreover, there are infinitely many NMS vector fields $\{X\}$ with $\mathcal{I}(X) = 0$ in the list: all NMS vector fields in case 1 and case 3 of the list.

Remark 1.4.

- Notice that in the last section of [8], the author has shown that all NMS vector fields with two periodic orbits are topologically equivalent. On the other hand, Dufraine [1] has computed that $\mathcal{I}(X) = 0$ when X is a NMS vector field on S^3 with two periodic orbits.
- Unfortunately, there still exist eight NMS vector fields (see the last section of [8]) with three periodic orbits which we do not know how to compute their homotopy numbers. Our algorithm strongly depends on the property that some NMS flows can be projected to some very simple vector fields on S^2. But it seems that it is not easy to construct such kind of projections for the exceptional eight NMS vector fields.
- Nevertheless, combining the results and comments above with Theorem 1.3, we have nearly answered Question 1.2. What we left just is to compute the homotopy numbers of the exceptional eight NMS vector fields.

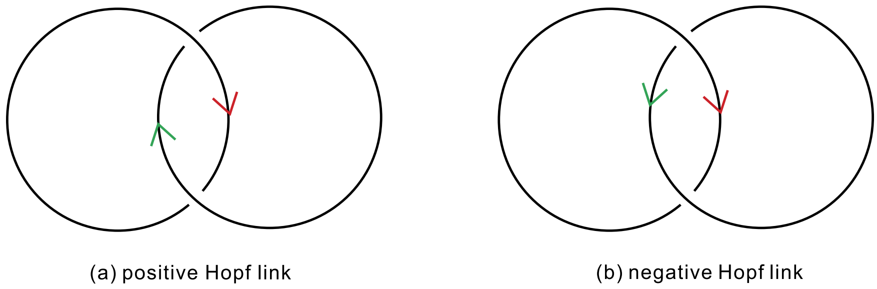

Fig. 1. (Color online.) Hopf link.

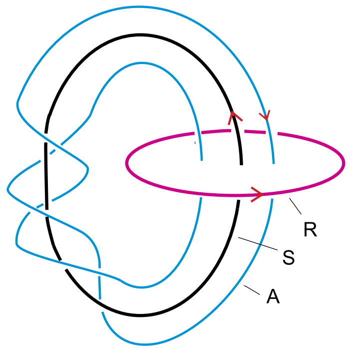

Fig. 2. (Color online.) The link type of the periodic orbits in the case $p = 2$ of case 2.

2. Preliminaries

2.1. Nonsingular Morse–Smale vector fields on S^3 with 2 and 3 periodic orbits

Definition 2.1. A smooth vector field ϕ_t is called a nonsingular Morse–Smale vector field (abbreviated as a NMS vector field) if it satisfies the following conditions:

1. the non-wandering set of ϕ_t is composed of finitely many periodic orbits without singularity;
2. each periodic orbit of ϕ_t is hyperbolic, i.e., the Poincaré return map for each periodic orbit is hyperbolic;
3. the stable and unstable manifolds of periodic orbits intersect transversally.

More details about the following facts in this subsection can be found in Section 5 of [8].

On S^3, up to topological equivalence, only one NMS vector field X whose periodic orbits are composed of an attractor A and a repeller R. Moreover, the periodic orbits $A \sqcup R$ form a Hopf link in S^3. More subtly, we can call such a NMS vector field X_+ (*resp.*, X_-) with positive (*resp.*, negative) Hopf link. For positive and negative Hopf links, see Fig. 1. X_+ and X_- are topologically equivalent but every topologically equivalent homeomorphism is not isotopic to the identity map.

The periodic orbits set of a NMS vector field X with 3 periodic orbits is composed of an attractor A, a repeller R and a saddle periodic orbit S. X has the following two possibilities. One case is that S is a normal saddle periodic orbit, X is one of eight NMS vector fields (see [8]). As we have mentioned in the last section, we do not deal with this case in this paper. The other case is that S is a twisted saddle periodic orbit. In this case, S and one of A and R form a positive Hopf link. The other one of A and R is a $(2, 2p-1)$ ($p \in \mathbb{Z}$) torus knot. The two parameters in the following list completely represent and decide the topologically equivalent class. Moreover, the vector fields in the list are pairwise different (up to topological equivalence).

1. $L(A, S) = 2p - 1$ ($p \in \mathbb{Z}$) and $L(R, S) = 1$. Here $L(A, S)$ is the linking number of A and S. Moreover, A is in the same direction of S in the complement of R.
2. $L(A, S) = 2p - 1$ ($p \in \mathbb{Z}$) and $L(R, S) = 1$. A is in the opposite direction of S in the complement of R.
3. $L(R, S) = 2p - 1$ ($p \in \mathbb{Z}$ and $p \neq 0, 1$) and $L(A, S) = 1$. R is in the same direction of S in the complement of A.
4. $L(R, S) = 2p - 1$ ($p \in \mathbb{Z}$ and $p \neq 0, 1$) and $L(A, S) = 1$. R is in the opposite direction of S in the complement of A.

Fig. 2 shows the link type of the periodic orbits of the NMS vector field associated with the case $p = 2$ of case 2.

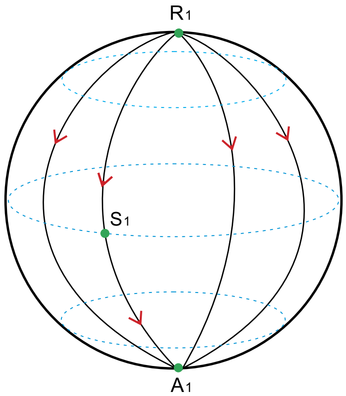

Fig. 3. (Color online.) The vector field Y on S^2.

2.2. An algorithm for computing the homotopy number of NMS vector fields on S^3

In this subsection, we will recall some tools developed by Dufraine in [1] to compute $\mathcal{I}(X)$ for a given NMS vector field X on S^3.

Given $p, q \in \mathbb{Z} - \{0\}$ such that p and q are coprime, a (p, q) Seifert fiberation map $\mathcal{S}_{p,q}$ of S^3 is a map from $S^3 \subset \mathbb{C}^2$ to $S^2 \subset \mathbb{C}P^1$ defined by: $\mathcal{S}_{p,q} = (z_1, z_2) \to [\frac{z_2^q}{z_1^p}]$.

A Seifert fiberation map $\mathcal{S}_{p,q}$ gives S^3 a Seifert fiber structure. We denote by $H_{p,q}$ the unit tangent vector field to the fibers of the (p, q) Seifert fiberation. In particular, we label $H_{1,1}$ and $H_{-1,1}$ by $\mathcal{H}_+$ and $\mathcal{H}_-$ correspondingly.

Suppose that X and Y are two nonsingular vector fields on S^3, we define $C_+(X, Y)$ and $C_-(X, Y)$ (sometimes simplified as C_+ and C_- correspondingly) as follows:

$$C_+(X, Y) = \{x \in S^3 \mid X(x) = \lambda Y(x), \lambda > 0\},$$

$$C_-(X, y) = \{x \in S^3 \mid X(x) = \lambda Y(x), \lambda < 0\}.$$

We have the following facts. More details can be found in [1].

1. The vector field $H_{p,q}$ is homotopic to $\mathcal{H}_+$ if $pq > 0$, and to $\mathcal{H}_-$ if $pq < 0$. X_+ is homotopic to $\mathcal{H}_+$ and X_- is homotopic to $\mathcal{H}_-$ correspondingly.
2. $\mathcal{I}(\mathcal{H}_+) = \mathcal{I}(\mathcal{H}_-) = 0$.
3. Generically, C_+ and C_- are two disjoint links in S^3. The distance between C_+ and C_-, $\mathcal{D}(C_+, C_-)$ is defined as the absolute value of the sum of the link numbers of all two component links between C_+ and C_-. To compute link numbers, we need to endow C_+ and C_- with suitable orientations. In particular, if C_+ and C_- are orbits of X (or of Y), all their components are oriented as orbits of X (or of Y).
4. $D(C_+, C_-)$ is a homotopy invariant. More precisely, if X' is homotopic to X, $C'_+ = C_+(X', Y)$ and $C'_- = C_-(X', Y)$ are two links in S^3, then $D(C_+, C_-) = D(C'_+, C'_-)$.
5. $\mathcal{I}(X) = \frac{\mathcal{D}(X,\mathcal{H}_+)+\mathcal{D}(X,\mathcal{H}_-)-1}{2}$.

3. Proof of the main theorem

Let's return to our main topic, i.e., NMS vector fields on S^3. In each topologically equivalent class of NMS vector fields with three periodic orbits on S^3 such that one periodic orbit is a twisted saddle periodic orbit, we always can choose a vector field X such that under some Seifert fiberation map $\mathcal{S} : S^3 \to S^2$, X induces a vector field Y on S^2 with three singularities R_1, S_1 and A_1 corresponding to R, S and A respectively. Y is as Fig. 3 shows.

Proof. We only need to compute $\mathcal{I}(X)$ for every case in the list of NMS vector fields with 3 periodic orbits in Section 2.1.

- If X is a vector field in case 1 and case 3 of the list, it is easy to prove that X is either homotopic to $\mathcal{H}_+$ or $\mathcal{H}_-$. Therefore, $\mathcal{I}(X) = 0$.
- If X is a vector field for some $p > 0$ in case 2 of the list.
  In this case, $H_{2,2p-1}$ is homotopic to $\mathcal{H}_+$ and $H_{-2,2p-1}$ is homotopic to $\mathcal{H}_-$. Under the Seifert fiberation map $\mathcal{S} : S^3 \to S^2$, $H_{2,2p-1}$ and $H_{-2,2p-1}$ induce the trivial vector fields on S^2 in the sense that every point in S^2 is a singular point. Notice that Y induced by X and $\mathcal{S}$ is a vector field with exactly three singularities R_1, S_1 and A_1. Therefore,

1. $C_+(X, H_{2,2p-1}) = \{R, S\}$ and $C_-(X, H_{2,2p-1}) = \{A\}$;
2. $C_+(X, H_{-2,2p-1}) = \{R, A\}$ and $C_-(X, H_{2,2p-1}) = \{S\}$.

Notice that

$$L(S, A) = 2p - 1, \qquad L(S, R) = 1 \quad \text{and} \quad L(A, R) = -2.$$

Then

$$\mathcal{D}(X, \mathcal{H}_+) = \mathcal{D}(X, H_{2,2p-1}) = \big|L(S, A) + L(R, A)\big| = |2p - 3|,$$

and

$$\mathcal{D}(X, \mathcal{H}_-) = \mathcal{D}(X, H_{-2,2p-1}) = \big|L(A, S) + L(R, S)\big| = |2p - 1 + 1| = 2|p|.$$

Therefore

$$\mathcal{I}(X) = \frac{\mathcal{D}(X, \mathcal{H}_+) + \mathcal{D}(X, \mathcal{H}_-) - 1}{2} = \frac{1}{2}\big(|2p - 3| + 2|p| - 1\big).$$

This means that

$$\mathcal{I}(X) = \begin{cases} 1, & \text{if } p = 1, \\ 2(p-1), & \text{if } p > 1. \end{cases}$$

- If X is a vector field for some $p \leqslant 0$ in case 2 of the list. In this case, $H_{2,2p-1}$ is homotopic to $\mathcal{H}_-$ and $H_{-2,2p-1}$ is homotopic to $\mathcal{H}_+$. By the similar reason as above, we have:
  1. $C_+(X, H_{2,2p-1}) = \{S\}$ and $C_-(X, H_{2,2p-1}) = \{R, A\}$;
  2. $C_+(X, H_{-2,2p-1}) = \{A\}$ and $C_-(X, H_{2,2p-1}) = \{R, S\}$.

  Notice that

$$L(S, A) = 2p - 1, \qquad L(S, R) = 1 \quad \text{and} \quad L(A, R) = -2.$$

  Then

$$\mathcal{D}(X, \mathcal{H}_+) = \mathcal{D}(X, H_{2,2p-1}) = \big|L(S, R) + L(S, A)\big| = 2|p|,$$

  and

$$\mathcal{D}(X, \mathcal{H}_-) = \mathcal{D}(X, H_{-2,2p-1}) = \big|L(A, R) + L(A, S)\big| = |2p - 3|.$$

  Therefore

$$\mathcal{I}(X) = \frac{\mathcal{D}(X, \mathcal{H}_+) + \mathcal{D}(X, \mathcal{H}_-) - 1}{2} = \frac{1}{2}\big(2|p| + |2p - 3| - 1\big) = 1 - 2p.$$

- For $p > 1$ and $p < 0$ in case 4 of the list, one can do similar computations as above to obtain that

$$\mathcal{I}(X) = \begin{cases} 2(p-1), & \text{if } p > 1, \\ 1 - 2p, & \text{if } p < 0. \end{cases}$$

In summary, we can collect these results to the statements of the theorem. □

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