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Schwarz method justification of a coupling between finite elements and integral representation for Maxwell exterior problem



Justification par la méthode de Schwarz du couplage entre éléments finis et représentation intégrale pour le problème de Maxwell en domaine extérieur

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ABSTRACT

We are interested in the resolution of an exterior Maxwell problem in 3D using a coupling between finite elements and integral representation. This strategy is interpreted as a Schwarz method which suggests a preconditioner for Krylov solvers. Numerical results confirm the relevance of the resolution scheme.

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RÉSUMÉ

Nous nous intéressons à la résolution numérique du problème de Maxwell extérieur en 3D par la méthode de couplage entre éléments finis et représentations intégrales. Nous limitons notre étude au cas d'un obstacle de type conducteur parfait. Cette stratégie est interprétée comme une méthode de Schwarz qui suggère un préconditionneur pour les solveurs de Krylov. Par suite, nous menons une étude numérique pour valider la pertinence d'un tel choix.

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Nous nous intéressons au problème de diffraction d'ondes électromagnétiques par un conducteur parfait en régime harmonique. Pour résoudre ce problème extérieur, nous considérons la méthode de couplage entre éléments finis et représentations intégrales (problème (2), connue sous la dénomination CEFRI, cf. [3]). Dans ce papier, nous analysons plus spécifiquement une stratégie de résolution itérative du couplage CEFRI inspirée de [4]. Nous montrons dans un premier temps l'analogie entre la méthode CEFRI et la méthode de Schwarz avec recouvrement total pour la résolution du problème de Maxwell en domaine non borné (problèmes (5) et (6)). Cette réinterprétation de la méthode CEFRI a été initialement

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Fig. 1. Left: exterior unbounded domain Ω_e , right: bounded domain Ω .

présentée dans [1]. Dans le cas d'une géométrie sphérique, elle nous permet d'établir le taux de convergence de la méthode itérative inhérente au problème (2) : la méthode de Schwarz avec recouvrement total converge si le bord Γ de l'obstacle est assez éloigné de la frontière artificielle Σ . Ce résultat est confirmé par des tests numériques réalisés avec des épaisseurs différentes entre Γ et Σ (voir Fig. 2). Cette relecture de la méthode CEFRI offre également une technique de préconditionnement.

Pour des raisons pratiques de mise en œuvre numérique, la prise en compte de la condition essentielle imposée sur le bord Γ se fait par une stratégie de pénalisation qui consiste à remplacer la condition au bord $E \times n_\gamma = 0$ du problème (2) par $\varepsilon_p (n_\gamma \times \text{curl} E) + E \times n_\gamma = 0$ où $\varepsilon_p > 0$ est choisi petit. Après discrétisation, cette méthode nous amène à résoudre un système linéaire impliquant une matrice éléments finis classique A ainsi qu'un bloc plein C dû à la représentation intégrale. La matrice du système linéaire à résoudre $A + C$ est mal conditionnée. En appliquant le préconditionneur suggéré par la méthode de Schwarz sur l'équation induite de la discrétisation du problème de diffraction (3), le système devient alors

$$(I + A^{-1}C)E = A^{-1}F, \quad (1)$$

F étant le second membre du système initial.

Un travail numérique de développement de nouveaux outils intégraux dans la librairie éléments finis MÉLINA++ [5] a permis la validation de la méthode CEFRI et d'une stratégie itérative de type Krylov. Plusieurs cas tests montrent la convergence éléments finis ainsi que la convergence de la résolution itérative : La Fig. 3 (gauche) montre l'erreur relative pondérée par la matrice du système linéaire en fonction de la finesse du maillage pour différentes valeurs du paramètre de pénalisation à un nombre d'onde fixé ; La Fig. 3 (droite) montre la convergence du solveur itératif pour différents nombres d'onde.

1. Introduction

We are interested in the resolution of the 3D exterior time-harmonic Maxwell equations. To this aim, we focus on a combination of finite elements and integral representation [3]. This strategy leads to an equivalent problem on a reduced bounded domain delimited by the surface of the scatterer and an artificial boundary with exact artificial boundary condition. No *a priori* condition is required on the distance between the scatterer and the artificial boundary but a difficult issue consists in the elaboration of a resolution strategy. A relevant idea was suggested in [4]. We propose the interpretation of this idea as an application of the Schwarz method, following the work done in [1] for Helmholtz equation. Hence, the theory on the Schwarz method justifies the use of Krylov solvers and the choice of a preconditioner.

In the next section, we derive the formulation of the system to be solved. Section 3 is devoted to the Schwarz interpretation of the resolution strategy suggested in [4]. In Section 4, the speed of convergence is estimated in the case of a spherical scatterer. Finally, some numerical results illustrate the theoretical properties.

2. Scattering by a perfect conductor

Let us consider Ω_i a bounded scatterer in \mathbb{R}^3 with a regular boundary Γ and Ω_e its unbounded complementary. We are concerned with the scattering of a time-harmonic electromagnetic wave by the perfect conductor Ω_i . Our purpose is to determine the total field $E = E^s + E^{inc}$ where E^{inc} is the incident wave and E^s is the scattered field, solution to the regularized Maxwell equations with essential boundary condition on Γ and radiation condition at infinity. Considering an integral representation on an artificial boundary Σ (see Fig. 1), the exterior problem reduces to a problem on a bounded domain Ω delimited by Γ and Σ (see [3]): Find E such that

$$\begin{cases} \text{curl curl } E - t^{-1} \nabla(\text{div } E) - k_s^2 E = 0 & \text{in } \Omega, \\ E \times n_\gamma = 0, \quad \text{div } E = 0 & \text{on } \Gamma, \\ T_{v_1}(E) = T_{v_1}(E^{inc} - \mathcal{I}_\Gamma(E)) \quad \text{and} \quad N_{v_2}(E) = N_{v_2}(E^{inc} - \mathcal{I}_\Gamma(E)) & \text{on } \Sigma, \end{cases} \quad (2)$$

where n_γ is the exterior unit normal of the domain Ω_i on Γ ; the regularization term $t^{-1} \nabla(\text{div } E)$ allows the use of a Galerkin finite-elements method (see [3]) and the regularization parameter t^{-1} depends on the permittivity and the permeability of the air; k_s is the wave number; v_1 and v_2 are complex numbers which have a negative imaginary part. The two operators T_{v_1} and N_{v_2} are defined by $T_{v_1} E = \text{curl } E \times n_\sigma + v_1 n_\sigma \times (E \times n_\sigma)$ and $N_{v_2} E = \text{div } E + v_2 E \cdot n_\sigma$ with n_σ the exterior unit normal of the domain Ω on Σ . The boundary conditions on Σ are derived from the integral representations satisfied by the scattered field and identified by the following expression [3]: for $x \in \Omega_e$,

$$\begin{aligned} \mathcal{I}_\Gamma(E)(x) = & -k_s^2 \int_{\Omega} \mathcal{R}\mathcal{G}_t(x, \cdot) E + \int_{\Omega} \operatorname{curl} \mathcal{R}\mathcal{G}_t(x, \cdot) \operatorname{curl} E + t^{-1} \int_{\Omega} \operatorname{div} \mathcal{R}\mathcal{G}_t(x, \cdot)^T \operatorname{div} E \\ & - t^{-1} \int_{\Gamma} \operatorname{div} \mathcal{G}_t(x, \cdot)^T (E \cdot n_\gamma) d\gamma, \end{aligned}$$

where $\mathcal{G}_t = G_{k_s} I + \frac{1}{k_s^2} \operatorname{Hess}(G_{k_s} - G_{k_p})$ is the outgoing Green tensor associated with the differential operator $\operatorname{curl} \operatorname{curl} - t^{-1} \nabla(\operatorname{div}) - k_s^2 I$ of the regularized Maxwell equation; I is the identity matrix in \mathbb{R}^3 ; Hess stands for Hessian operator; $k_p = \sqrt{t} k_s$ and G_k is the fundamental solution of Helmholtz equation; \mathcal{R} is a linear operator that maps every regular function φ defined on Γ into a regular function $\mathcal{R}\varphi$ defined on Ω which satisfies $\mathcal{R}\varphi = \varphi$ on Γ and $\mathcal{R}\varphi = 0$ on Σ . The consideration of the Hilbert space

$$\mathcal{H}_t = \{E \in H(\operatorname{curl}, \Omega) / \operatorname{div} E \in L^2(\Omega), E \times n_\gamma = 0 \text{ on } \Gamma, E \times n_\sigma \in L^2(\Sigma)^3, E \cdot n_\sigma \in L^2(\Sigma)\},$$

enables one to write a variational formulation of the problem (2): Find $E \in \mathcal{H}_t$ such that

$$(\mathcal{A}_t + \mathcal{C}_t)E = F_t, \tag{3}$$

where the operators \mathcal{A}_t and $\mathcal{C}_t : \mathcal{H}_t \rightarrow \mathcal{H}_t$ are defined as follows

$$\begin{aligned} (\mathcal{A}_t E, E')_t = & \int_{\Omega} (\operatorname{curl} E \cdot \operatorname{curl} E' + t^{-1} \operatorname{div} E \operatorname{div} E' - k_s^2 E \cdot E') \\ & + \int_{\Sigma} (v_1(n_\sigma \wedge E) \cdot (n_\sigma \wedge E') + t^{-1} v_2(n_\sigma \cdot E)(n_\sigma \cdot E')) d\sigma, \\ (\mathcal{C}_t E, E')_t = & \int_{\Sigma} T_{v_1}(\mathcal{I}_\Gamma(E)) \cdot E' d\sigma + t^{-1} \int_{\Sigma} N_{v_2}(\mathcal{I}_\Gamma(E))(n_\sigma \cdot E') d\sigma, \end{aligned}$$

and F_t is given by $(F_t, E')_t = \int_{\Sigma} (T_{v_1}(E^{inc}) \cdot E' + t^{-1} N_{v_2}(E^{inc})(n_\sigma \cdot E')) d\sigma$, where $(\cdot, \cdot)_t$ is the scalar product on \mathcal{H}_t .

The problem (3) is well posed as explained in [3] and the operator \mathcal{A}_t is invertible.

3. Schwarz method interpretation

In order to solve the system (3), Jin and Liu [4] suggested to consider \mathcal{C}_t on the right hand side. An application of the fixed point algorithm leads to finding E_{n+1} such that

$$\begin{cases} \operatorname{curl} \operatorname{curl} E_{n+1} - t^{-1} \nabla(\operatorname{div} E_{n+1}) - k_s^2 E_{n+1} = 0 & \text{in } \Omega, \\ E_{n+1} \times n_\gamma = 0, \quad \operatorname{div} E_{n+1} = 0 & \text{on } \Gamma, \\ T_{v_1}(E_{n+1}) = T_{v_1}(E^{inc} - \mathcal{I}_\Gamma(E_n)) \quad \text{and} \quad N_{v_2}(E_{n+1}) = N_{v_2}(E^{inc} - \mathcal{I}_\Gamma(E_n)) & \text{on } \Sigma. \end{cases} \tag{4}$$

In this paper, we focus on an original mathematical justification of convergence of the algorithm expressed by Jin and Liu. We interpret the algorithm defined by (4) as a Schwarz method. This interpretation has been initially proposed for the case of Helmholtz equation in [1]. The strategy is designed by the Total Overlapping Schwarz method. Indeed the overlapping area is the total domain Ω . We hereby extend their work to the case of Maxwell equations: it consists in replacing equivalently the problem (4) by the two following subproblems. The first one is a transmission problem:

$$\begin{cases} \operatorname{curl} \operatorname{curl} E_{2n+1} - t^{-1} \nabla(\operatorname{div} E_{2n+1}) - k_s^2 E_{2n+1} = 0 & \text{in } \Omega_i \cup \Omega_e, \\ n_\gamma \times [E_{2n+1}] = 0, \quad n_\gamma \times [\operatorname{curl} E_{2n+1}] = -n_\gamma \times \operatorname{curl} E_{2n} & \text{on } \Gamma, \\ [\operatorname{div} E_{2n+1}] = 0, \quad n_\gamma \cdot [E_{2n+1}] = -n_\gamma \cdot E_{2n} & \text{on } \Gamma, \\ \lim_{\rho \rightarrow \infty} \int_{\|x\|=\rho} \|\operatorname{curl} E_{2n+1}^s \times n_\sigma - ik_s n_\sigma \times (E_{2n+1}^s \times n_\sigma)\|^2 d\sigma = 0, \\ \lim_{\rho \rightarrow \infty} \int_{\|x\|=\rho} |\sqrt{t^{-1}} \operatorname{div} E_{2n+1}^s - ik_s E_{2n+1}^s \cdot n_\sigma|^2 d\sigma = 0. \end{cases} \tag{5}$$

The second one consists in finding E_{2n+2} such that

$$\begin{cases} \operatorname{curl} \operatorname{curl} E_{2n+2} - t^{-1} \nabla(\operatorname{div} E_{2n+2}) - k_s^2 E_{2n+2} = 0 & \text{in } \Omega, \\ E_{2n+2} \times n_\gamma = 0, \quad \operatorname{div} E_{2n+2} = 0 & \text{on } \Gamma, \\ T_{v_1}(E_{2n+2}) = T_{v_1}(E_{2n+1}) \quad \text{and} \quad N_{v_2}(E_{2n+2}) = N_{v_2}(E_{2n+1}) & \text{on } \Sigma. \end{cases} \tag{6}$$

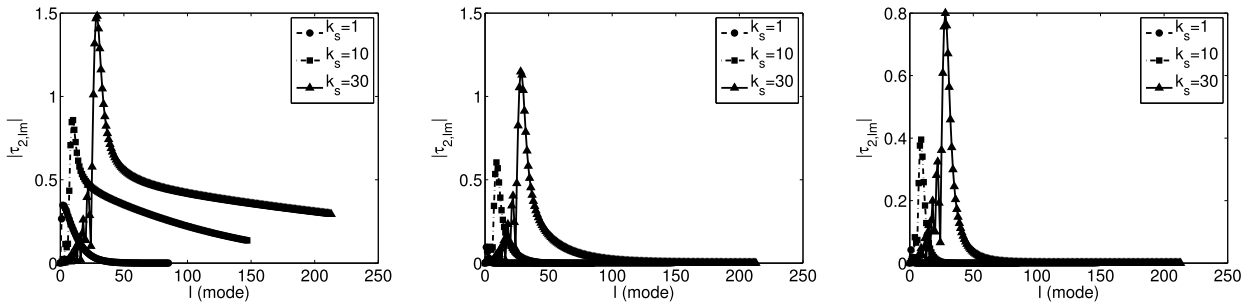


Fig. 2. Modulus of $\tau_{2,lm}$ for thickness $e = \lambda/100$ (left), $\lambda/10$ (center) and $\lambda/5$ (right). Cases $k_s = 1, 10$ or 30 .

The solution E_{2n+1} of (5) has an explicit solution given by an integral representation. By inserting this representation in the second condition of (6) we effectively obtain the solution of (4). At the iteration n , the Schwarz algorithm is defined by $\mathcal{A}_t E_{n+1} = -\mathcal{C}_t E_n + F_t$. Numerically, we use the scheme suggested by Jin and Liu and do not use the subproblems (5) and (6). The intermediate problems (5) and (6) are used for theoretical justifications. This enables one to derive convergence estimations that cannot be obtained directly from the system (4). In Section 4, we investigate an analytical calculation of the rate of convergence of the Schwarz method in a spherical configuration.

4. Analytical estimation of the convergence for a spherical scatterer

In the case where Ω_i is a perfectly conducting ball, we investigate the rate of convergence of the Total Overlapping Schwarz method. Let us consider the scatterer to be a ball of radius R_* . We suppose that the artificial boundary Σ is a sphere concentric to Γ with radius $R > R_*$. We first introduce some notations: We denote by j_l the spherical Bessel function of degree l , by h_l the spherical Hankel function of the first kind of degree l and $H_l(r) = h_l(r) + rh_l'(r)$, $J_l(r) = j_l(r) + rj_l'(r)$. We introduce the tangential vector spherical harmonics on the unit sphere S_1 , $U_{lm} = (1/\sqrt{l(l+1)})\nabla Y_l^m$ and $V_{lm} = n_\gamma \times U_{lm}$, where Y_l^m , $l > 0$, $m = -l, \dots, l$ are the orthonormal scalar spherical harmonics (complete basis of $L^2(S_1)$). The sets of U_{lm} and V_{lm} form a complete orthonormal basis for $T^2(S_1) := \{a : S_1 \rightarrow \mathbb{C}^3/a \in (L^2(S_1))^3, a \cdot n_\gamma = 0\}$.

We define the error $(w_n)_n$ on the field E at each iteration by:

$$w_{2n+1} = E_{2n+1} - E \text{ in } \Omega_e, \quad w_{2n+1} = E_{2n+1} \text{ in } \Omega_i, \quad w_{2n+2} = E_{2n+2} - E \text{ in } \Omega.$$

For all n , we define $\Lambda_n = T_{v_1}(w_{2n})$ and $\delta_n = N_{v_2}(w_{2n})$. In order to prove that the error $(w_n)_n$ converges to zero, we first show that $\Lambda_{n+1} = K\Lambda_n$ and $\delta_{n+1} = L\delta_n$ with $K : T^2(S_1) \rightarrow T^2(S_1)$ and $L : L^2(S_1) \rightarrow L^2(S_1)$ two linear maps. K (resp. L) has a diagonal representation in the basis $(U_{lm}, V_{lm})_{lm}$ of $T^2(S_1)$ (resp. Y_l^m of $L^2(S_1)$). Let us denote by $\tau_{1,lm}, \tau_{2,lm}$ (resp. $\tau_{3,lm}$) the eigenvalues of K (resp. L). These eigenvalues define the rate of convergence of the Total Overlapping Schwarz method. Taking into account the boundary and transmission conditions, we obtain:

$$\tau_{1,lm} = \left(1 - \frac{H_l(k_s R_*)}{J_l(k_s R_*)} \frac{J_l(k_s R) - ik_s R j_l(k_s R)}{H_l(k_s R) - ik_s R h_l(k_s R)}\right)^{-1},$$

$$\tau_{2,lm} = \left(1 - \frac{h_l(k_s R_*)}{j_l(k_s R_*)} \frac{J_l(k_s R) - ik_s R j_l(k_s R)}{H_l(k_s R) - ik_s R h_l(k_s R)}\right)^{-1}, \quad \tau_{3,lm} = \left(1 - \frac{H_l(k_s R_*)}{J_l(k_s R_*)} \frac{j_l(k_s R)}{h_l(k_s R)}\right)^{-1}.$$

The convergence of the Total Overlapping Schwarz method is ensured if $|\tau_{i,lm}| < 1, \forall i = 1, \dots, 3, \forall l$. The reader can see that these eigenvalues are independent of the parameter m . For $R_* = 1$, the asymptotic behavior of the spherical Bessel functions for large l leads to the asymptotic estimation $\tau_{i,lm} \sim (1 - R^{2l})^{-1}, i = 1, \dots, 3$. As a consequence, for small values of R , there exists a finite number of coefficients $\tau_{i,lm}$ outside of the unit disk, and for sufficiently large values of R , all the coefficients $\tau_{i,lm}$ are in the interior to the unit disk. We conclude that the Schwarz method has a linear convergence for R large enough. The numerical tests illustrate this theoretical result. In Fig. 2, we consider $R_* = 1$ and $R = R_* + e$ with different values of $e : \lambda/100, \lambda/10$ or $\lambda/5$. The cases $e = \lambda/100$ and $e = \lambda/10$ exhibit some coefficients larger than 1 while the maximum value of $|\tau_{2,lm}|$ is strictly lower than 1 at the considered wavenumbers for the thickness $e = \lambda/5$ but the results are strongly dependent on the wavenumber. Similar asymptotic observations can be done on $|\tau_{1,lm}|$ and $|\tau_{3,lm}|$. As a consequence, a Krylov method is a relevant alternative to the algorithm defined by (4): due to the properties of Krylov solvers demonstrated in [2], the convergence of a Krylov method is ensured for the resolution of the problem (3) using \mathcal{A}_t as a preconditioner.

5. Preconditioner for a Krylov solver

The previous work suggests the use of \mathcal{A}_t as a preconditioner to solve the problem (3) using a Krylov solver. We hereby consider the resolution of problem (3) using the biconjugate gradient stabilized method. After a finite-elements

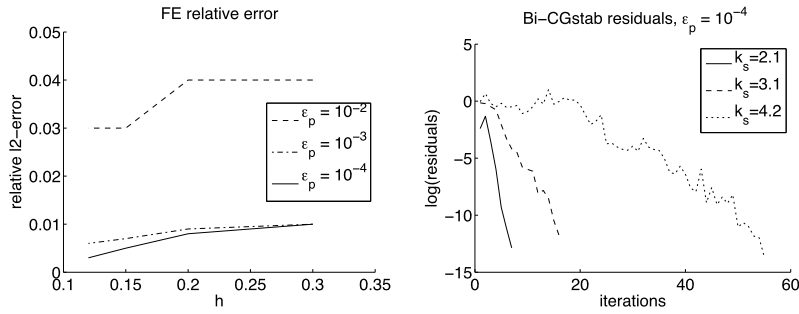


Fig. 3. Relative l^2 -error with respect to the discretization (left), behavior of the residuals, case $k_s h = 2\pi/10$ (right).

discretization, the linear system is written under the form $(A + C)E = F$ and the preconditioned system becomes $(I + A^{-1}C)E = A^{-1}F$ where the matrix C involves the integral operators and the matrix A related to the differential operators involves a term resulting from the essential condition considered by a penalization strategy: $\varepsilon_p (n_\gamma \times \text{curl } E) + E \times n_\gamma = 0$ on Γ , with $\varepsilon_p > 0$.

The numerical implementation were done using and developing new integrands in the library MÉLINA++ [5]. In this section, we validate the efficiency of the preconditioning strategy by the consideration of an intermediate problem the solution of which is known:

$$\left\{ \begin{array}{l} \text{curl curl } E - \nabla(\text{div } E) - k_s^2 E = 0 \quad \text{in } \Omega_e, \\ E \times n_\gamma = \mathcal{G}_1^1 \times n_\gamma, \quad \text{div } E = \text{div } \mathcal{G}_1^1 \quad \text{on } \Gamma, \\ \lim_{\rho \rightarrow \infty} \int_{\|x\|=\rho} \|\text{curl } E^s \times n_\sigma - ik_s n_\sigma \times (E^s \times n_\sigma)\|^2 d\sigma = 0, \\ \lim_{\rho \rightarrow \infty} \int_{\|x\|=\rho} |\text{div } E^s - ik_s E^s \cdot n_\sigma|^2 d\sigma = 0, \end{array} \right. \quad (7)$$

where \mathcal{G}_1^1 is the first vector component of \mathcal{G}_1 . The scatterer is the unit sphere and the artificial boundary Σ is the sphere concentric to Γ with radius $R = 1.5$.

Fig. 3-left shows the convergence of the relative error with respect to the finite-elements discretization: the relative error is plotted with respect to the average size of the mesh elements, for different values of the penalization parameter ε_p , with $k_s = 3$. Fig. 3-right illustrates a superlinear convergence of the biconjugate gradient stabilized solver applied to the preconditioned system for different values of the wavenumber with $\varepsilon_p = 10^{-4}$.

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