



Number theory/Mathematical analysis

Note on the differentiability of arithmetic Fourier series arising from Eisenstein series



Note sur la dérivabilité de séries arithmétiques de Fourier provenant des séries d'Eisenstein

Izabela Petrykiewicz

Université Joseph-Fourier, Institut Fourier, 100, rue des Maths, 38402 Saint-Martin-d'Hères, France

ARTICLE INFO

Article history:

Received 25 November 2013

Accepted after revision 18 February 2014

Available online 11 March 2014

Presented by Jean-Pierre Kahane

ABSTRACT

The aim of this note is to present results concerning the differentiability of some Fourier series arising from Eisenstein series. Sine series exhibit different behaviours with respect to differentiability than the series with cosine function. The precise results are given for the series related to Eisenstein series of weight 2, whereas for the series arising from Eisenstein series of higher weight we conjecture the results.

© 2014 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

RÉSUMÉ

Le but de cette note est de présenter des résultats concernant la dérivabilité de certaines séries de Fourier découlant des séries d'Eisenstein. Les séries de sinus se comportent différemment des séries de cosinus. Les résultats précis sont donnés pour les séries liées à la série d'Eisenstein de poids 2. Pour les séries découlant des séries d'Eisenstein de poids supérieur à 2, nous formulons une conjecture.

© 2014 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction and statement of the results

In this note, we present some results concerning the differentiability of certain Fourier series related to Eisenstein series. Let $k \in \mathbb{N}^*$ be even. Recall that the Eisenstein series of weight k over the upper-half plane \mathbb{H} is defined by the first equality

$$E_k(z) = \frac{1}{2\zeta(k)} \sum_{\substack{m, n \in \mathbb{Z} \\ (m, n) \neq (0, 0)}} \frac{1}{(m + nz)^k} = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi inz},$$

E-mail address: izabela.petrykiewicz@ujf-grenoble.fr.

and the second equality provides its Fourier series, where B_k is the k -th Bernoulli number, and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. If $k = 2$, the function is quasi-modular under the action of $SL_2(\mathbb{Z})$, and for $k \geq 4$ it is modular. We are interested in the real-valued continuous functions F_k, G_k defined by

$$F_k(x) = \sum_{n=1}^{\infty} \frac{\sigma_{k-1}(n)}{n^{k+1}} \sin(2\pi nx) \quad \text{and} \quad G_k(x) = \sum_{n=1}^{\infty} \frac{\sigma_{k-1}(n)}{n^{k+1}} \cos(2\pi nx).$$

In particular, we focus on the differentiability of F_2 and G_2 . The differentiability of F_2 at x depends on the continued fraction expansion of x , but G_2 is probably differentiable at any $x \in \mathbb{R} \setminus \mathbb{Q}$. Already in 1933, Wilton in his work [7] proved that there is a connection between certain trigonometric series involving the divisor function $\sigma_0(n)$ and continued fractions.

Theorem 1.1. *Neither F_2 nor G_2 is differentiable at any $x \in \mathbb{Q}$. However, G_2 is right and left differentiable at each $x \in \mathbb{Q}$.*

Let $(\frac{p_n(x)}{q_n(x)})_{n \geq 0}$ be the sequence of continued fraction approximations of x , i.e. $\frac{p_n(x)}{q_n(x)} = [a_0; a_1, a_2, \dots, a_n]$. We then make the following definition, which is motivated by the work of Brjuno [1].

Definition 1.2. Let $x \in \mathbb{R} \setminus \mathbb{Q}$. We will say that x is a square-Brjuno number if $\sum_{n=0}^{\infty} \frac{\log q_{n+1}(x)}{q_n(x)^2} < \infty$.

In addition, we introduce two technical conditions:

$$\lim_{n \rightarrow \infty} \frac{\log q_{n+4}(x)}{q_n(x)^2} = 0; \tag{1}$$

$$\lim_{n \rightarrow \infty} \frac{\log q_{n+3}(x)}{q_n(x)^2} = 0, \quad \text{and} \quad a_n = 1 \text{ for only finitely many } n. \tag{2}$$

We note that the square-Brjuno property and conditions (1) and (2) are independent.

Theorem 1.3.

- (i) *If $x \in \mathbb{R} \setminus \mathbb{Q}$ is a square-Brjuno number satisfying (1) or (2), then F_2 is differentiable at x . If $x \in \mathbb{R} \setminus \mathbb{Q}$ is not a square-Brjuno number, then F_2 is not differentiable at x .*
- (ii) *Let $x \in \mathbb{R} \setminus \mathbb{Q}$ satisfy (1) or (2), then G_2 is differentiable at x .*

We observe that condition (1) is satisfied for almost all x , but that condition (2) holds for almost no x . We believe that both conditions (1) and (2) could be removed in Theorem 1.3; however, the applied method does not enable us to do this. Moreover, almost all numbers are square-Brjuno. If $x \in \mathbb{R} \setminus \mathbb{Q}$ is not square-Brjuno, then it must be Liouville. We prove Theorems 1.1 and 1.3 using the method proposed by Itatsu in the study of the Riemann “non-differentiable” function $S(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(\pi n^2 x)$, see [3].

2. Sketch of the proofs

Let φ_2 be the complex-valued function defined by $\varphi_2(x) = G_2(x) + iF_2(x)$. For a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $z \in \mathbb{C}$, we will denote the fraction transformation as $\gamma \cdot z = \frac{az+b}{cz+d}$, if $cz + d \in \mathbb{C} \setminus \{0\}$, and $\gamma \cdot (-\frac{d}{c}) = \infty$. Recall that for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and all $z \in \mathbb{H}$, we have $E_2(z) = E_2(\gamma \cdot z)(cz + d)^{-2} + 6ic\pi^{-1}(cz + d)^{-1}$. Based on this quasi-modular identity and the relationship between φ_2 and E_2 , we obtained the following functional equation for φ_2 .

Lemma 2.1. *Let $\frac{p}{q} \in \mathbb{Q}, \gamma = \begin{pmatrix} a & b \\ q & -p \end{pmatrix} \in SL_2(\mathbb{Z})$, and $x \in \mathbb{R}$. We have*

$$\begin{aligned} \varphi_2(x) &= (qx - p)^4 \varphi_2(\gamma \cdot x) - \frac{i\pi^3}{3q^3} (qx - p) \text{Log}(qx - p) + P_\gamma(qx - p) \\ &\quad - \frac{\pi^2}{q^2} (qx - p)^2 \text{Log}(qx - p) + 6 \int_{\frac{p}{q}}^x q(qt - p)^2 (q(x - t) - (qt - p)) \varphi_2(\gamma \cdot t) dt, \end{aligned} \tag{3}$$

where Log denotes the principal value of the complex logarithm and $P_\gamma \in \mathbb{C}[x]$ is a polynomial of degree less than or equal to 3 that depends on γ .

We then take the imaginary (or real) part of each side of Eq. (3). The conclusion of Theorem 1.1 follows from letting $x \rightarrow \frac{p}{q}$ and estimating the growth rates of each of the terms of (3). For F_2 , the principal term is given by $F_2(x) = F_2(\frac{p}{q}) - \frac{\pi^3}{3q^3}(qx - p) \log|qx - p|(1 + o(1))$, which explains that F_2 is not differentiable at $\frac{p}{q}$, whereas for G_2 the term $\text{Re}(-\frac{i\pi^3}{3q^3}(qx - p) \text{Log}(qx - p))$ determines the differentiability properties at $\frac{p}{q}$.

Let $T : [0, 1) \rightarrow [0, 1)$ denote the Gauss map, that is $T(0) = 0$ and $T(x) = \frac{1}{x} \bmod 1$ if $x \neq 0$. The function T is differentiable at all $x \notin \mathbb{Q}$. We apply (3) to the matrix $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Since φ_2 is 1-periodic, we have $\varphi_2(-\frac{1}{x}) = \varphi_2(-T(x))$. By taking the imaginary part of each side of the equation, we deduce from (3) the following equation.

Lemma 2.2. For all $x \in [0, 1)$ we have

$$F_2(x) = -x^4 F_2(T(x)) - \frac{\pi^3}{3} x \log(x) + P(x) - 6 \int_0^x t^2(x - 2t) F_2(T(t)) dt, \tag{4}$$

where $P \in \mathbb{R}[x]$ is of degree less than or equal to 3.

For brevity, write $\beta_{-1}(x) = 1$ and $\beta_n(x) = \prod_{j=0}^n T^j(x)$ for $n \geq 0$. Assume $x \notin \mathbb{Q}$. We then iterate Eq. (4) repeatedly substituting $T(x)$ for x to obtain the following.

Lemma 2.3. For all $x \in [0, 1) \setminus \mathbb{Q}$ and all $n \in \mathbb{N}^*$ we have

$$F_2(x) = (-1)^n F_2(T^n(x)) \beta_{n-1}(x)^4 + 6 \sum_{j=1}^n (-1)^j \beta_{j-2}(x)^4 \int_0^{T^{j-1}(x)} t^2(T^{j-1}(x) - 2t) F_2(T(t)) dt + \frac{\pi^3}{3} \sum_{j=1}^n (-1)^j \log(T^{j-1}(x)) \beta_{j-1}(x) \beta_{j-2}(x)^3 + \sum_{j=1}^n (-1)^{j-1} P(T^j(x)) \beta_{j-2}(x)^4, \tag{5}$$

where $P \in \mathbb{R}[x]$ is the same polynomial as in Lemma 2.2.

If $n \rightarrow \infty$, the three series in (5) converge for all $x \in \mathbb{R}$, and the term $(-1)^n F_2(T^n(x)) \beta_{n-1}(x)^4 \rightarrow 0$, which provides an alternative expression for F_2 .

We are interested in the limit $\lim_{h \rightarrow 0} \frac{F_2(x+h) - F_2(x)}{h}$. Therefore, for each $h > 0$, we choose a suitable $n_h \in \mathbb{N}$, which satisfies: $n_h \rightarrow \infty$ and is non-decreasing as $h \rightarrow 0$. We then apply Lemma 2.3 with $n = n_h$ and analyse the rate of change of each of the terms in (5) as $h \rightarrow 0$. The next lemma illustrates the origin of the square-Brjuno condition in Theorem 1.3.

Lemma 2.4. Write $f'(x)$ for the derivative of a function f at x . Given $x \in \mathbb{R} \setminus \mathbb{Q}$, the series

$$\frac{\pi^3}{3} \sum_{j=1}^{\infty} (-1)^j (\log(T^{j-1}(x)) \beta_{j-1}(x) \beta_{j-2}(x)^3)' \tag{6}$$

converges if and only if x is square-Brjuno. On the other hand, the two series

$$6 \sum_{j=1}^{\infty} (-1)^j \left(\beta_{j-2}(x)^4 \int_0^{T^{j-1}(x)} t^2(T^{j-1}(x) - 2t) F_2(T(t)) dt \right)' \text{ and } \sum_{j=1}^n (-1)^{j-1} (P(T^j(x)) \beta_{j-2}(x)^4)' \tag{7}$$

converge for all $x \in \mathbb{R} \setminus \mathbb{Q}$.

The technical conditions (1) and (2) arise from the analysis of the term

$$\frac{1}{h} ((-1)^{n_h} F_2(T^{n_h}(x+h)) \beta_{n_h-1}(x+h)^4 - (-1)^{n_h} F_2(T^{n_h}(x)) \beta_{n_h-1}(x)^4).$$

As $h \rightarrow 0$, it converges to 0 if either (1) or (2) is satisfied.

We follow the same strategy for G_2 ; however, we do not have an analogue of (6) which explains why we only need (1) or (2) in Theorem 1.3(ii). The detailed proofs of Theorems 1.1 and 1.3 will be published elsewhere [5]. If F_2 or G_2 is differentiable at x by Theorem 1.3, the value of the derivative is given by the addition of (6) and (7).

3. Generalisation

Conjecture 3.1. Let $k \in \mathbb{N}^*$ be even. We have the following.

- (i) Neither F_k nor G_k is differentiable at any $x \in \mathbb{Q}$. However, G_k is right and left differentiable at each $x \in \mathbb{Q}$.
- (ii) The function G_k is differentiable at any $x \in \mathbb{R} \setminus \mathbb{Q}$.
- (iii) The function F_k is differentiable at $x \in \mathbb{R} \setminus \mathbb{Q}$ if and only if

$$\sum_{n=0}^{\infty} \frac{\log q_{n+1}(x)}{q_n(x)^k} < \infty. \quad (8)$$

In order to prove [Conjecture 3.1](#) for $k \geq 4$, we would proceed as in the case $k = 2$. There are a lot of terms to analyse, but we believe that for any given $k \geq 4$, the method presented here would work (adding a technical condition similar to (1) of the type $\frac{\log(q_{n+4})}{q_n^k} \rightarrow 0$). However, the calculations become very long, and we do not do it explicitly. In [5], we present arguments justifying the conjecture.

Chamizo [2] studied the differentiability of series arising from automorphic forms $f(x) = \sum_{n=0}^{\infty} r_n e^{2\pi i n x}$ of positive weights k under the Fuchsian group with a multiplier system: $f_s(x) = \sum_{n=1}^{\infty} \frac{r_n}{n^s} e^{2\pi i n x}$. His method is based on the theory of automorphic forms. However, he requires $s < \frac{k}{2} + 1$ for f not being a cusp form. In our case, $s = k + 1$, which does not enable us to apply his results. Another method of analysing such series involves wavelet methods, and was proposed by Jaffard [4], again, in the study of $S(x)$. Studying the Hölder regularity exponent of F_k and G_k with this method enables us to prove some cases of [Conjecture 3.1](#). For each n , we define κ_n by the equality $|x - \frac{p_n}{q_n}| = \frac{1}{q_n^{\kappa_n}}$, and we let $\mu(x) = \limsup_{n \rightarrow \infty} \kappa_n$, $\nu(x) = \liminf_{n \rightarrow \infty} \kappa_n$. It has been proved in [6] that for $k \geq 4$ and $x \in \mathbb{R} \setminus \mathbb{Q}$, if $\frac{1}{\nu(x)} - \frac{1}{\mu(x)} < \frac{1}{k}$, then the Hölder regularity exponents of F_k and G_k at x are both $1 + \frac{k}{\mu(x)}$. If $\mu(x) < \infty$, then both F_k and G_k are differentiable at x . The condition $\mu(x) < \infty$ implies (8), and we see that one direction of [Conjecture 3.1](#)(iii) is true. Since for almost all x , $\mu(x) = \nu(x) = 2$, the conjecture is proved for almost all x for all $k \geq 4$.

References

- [1] A.D. Brjuno, Analytic form of differential equations. I, Tr. Mosk. Mat. Obs. 25 (1971) 119–262 (in Russian); A.D. Brjuno, Analytic form of differential equations. II, Tr. Mosk. Mat. Obs. 26 (1972) 199–239 (in Russian).
- [2] F. Chamizo, Automorphic forms and differentiability properties, Trans. Amer. Math. Soc. 356 (2004) 1909–1935.
- [3] S. Itatsu, Differentiability of Riemann's function, Proc. Jpn. Acad. Ser. A Math. Sci. 57 (10) (1981) 492–495.
- [4] S. Jaffard, The spectrum of singularities of Riemann's function, Rev. Mat. Iberoam. 12 (2) (1996) 441–460.
- [5] I. Petrykiewicz, Differentiability of Fourier series arising from Eisenstein series, in preparation.
- [6] I. Petrykiewicz, Hölder regularity of arithmetic Fourier series arising from modular forms, preprint, arXiv:1311.0655.
- [7] J.R. Wilton, An approximate functional equation with applications to a problem of Diophantine approximation, J. Reine Angew. Math. 169 (1933) 219–237.