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Numerical analysis

## A simple nonconforming quadrilateral finite element

*Un élément fini non conforme quadrilatéral simple*Boujemâa Achchab<sup>a</sup>, Abdellatif Agouzal<sup>b</sup>, Khalid Bouihat<sup>a</sup><sup>a</sup> LM2CE, LAMSAD, Univ. Hassan 1<sup>er</sup> FSJES and EST, B.P. 218, Berrechid, Morocco<sup>b</sup> Université de Lyon, CNRS, Université Lyon-1, Institut Camille-Jordan, 69622 Villeurbanne cedex, France

## ARTICLE INFO

## Article history:

Received 24 December 2013

Accepted after revision 13 March 2014

Available online 26 April 2014

Presented by Olivier Pironneau

## ABSTRACT

We introduce and analyze a simple nonconforming quadrilateral finite element and then we derive optimal a priori error estimates for arbitrary regular quadrilaterals. The idea of extension to some non-conforming elements for three-dimensional hexahedrons is also presented.

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## R É S U M É

Dans ce travail, nous présentons et analysons un élément fini non conforme en quadrangles. Nous obtenons une estimation d'erreur a priori optimale pour des quadrangles réguliers arbitraires. Nous présentons également l'idée d'extension tridimensionnelle de cet élément.

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## Version française abrégée

Les éléments finis non conformes jouissent de bonnes propriétés de stabilité ; ils ont intéressé de nombreux mathématiciens – voir par exemple [2,3,5,9,10]. En 1973, Crouzeix et Raviart [4] ont été les premiers à examiner l'élément triangulaire linéaire non conforme avec trois degrés de liberté situés au milieu des arêtes. Cet élément non conforme  $P_1-P_0$ , le plus simple, a été utilisé avec succès pour résoudre les équations de Stokes stationnaires. Ensuite, Han (1984) a proposé un élément rectangulaire non conforme pour résoudre les équations de Stokes stationnaires [7] et de Navier–Stokes [6]. Pour les équations de Stokes, Rannacher et Turek [11] ont introduit en 1992 un élément fini  $Q_1$ -non conforme sur les quadrilatères convexes.

Dans cette note, nous présentons et analysons un élément fini quadrilatère simple non conforme, pour lequel on donne une condition nécessaire et suffisante (2.4) pour l'unisolvance, et établissons des estimations d'erreur a priori optimales pour des quadrilatères réguliers arbitraires. Des choix particuliers des fonctions  $B_k$  définies plus loin nous permettent de retrouver des éléments finis classiques. Nous présentons également l'idée d'obtention des résultats analogues pour certains éléments non conformes en trois dimensions pour les hexahédres.

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## 1. Introduction

Crouzeix and Raviart [4] (1973) first considered the nonconforming linear *triangular* element with three nodes located at midpoints of edges. This simplest nonconforming  $P_1$ - $P_0$  element (piecewise linear elements for velocity and piecewise constants for pressure), which is called C–R triangular element today, has been successfully utilized for solving the stationary Stokes equations. Han (1984) proposed a nonconforming *rectangular* element for solving the stationary Stokes equations [6] and Navier–Stokes equations [7]. For the Stokes equations, Rannacher and Turek [11] (1992) introduced the so-called ‘rotated’  $Q_1$ -nonconforming finite element on arbitrarily convex quadrilaterals. The corresponding local finite element spaces are obtained by rotating the mixed term of the bilinear element, and the local degrees of freedom are either the average of the function over the edge or its value at the midpoint of the edge. Nonconforming finite elements enjoy better stability properties compared to the conforming finite elements [8]. They have attracted the attention of many mathematicians, see, e.g. [2,3,5,9,10,12].

In this note, we introduce and analyze a simple nonconforming quadrilateral finite element and derive optimal a priori error estimates for arbitrary regular quadrilaterals. The idea of extension to some non-conforming elements for three-dimensional hexahedrons is also presented.

## 2. A quadrilateral nonconforming element

Let  $K$  be a convex and non-degenerate quadrilateral domain with vertices  $\{a_i\}_{1 \leq i \leq 4}$  numbered counterclockwise. Denote by  $\{e_i\}_{1 \leq i \leq 4}$  its edges  $[a_i, a_{i+1}]$ , in which the indices are numbered modulo four. In order to introduce our nonconforming finite element, let us assume that we are given a function  $B_K \in C^0(K)$  and define the two sets:

$$R_K = P_1(K) + \text{span}\{B_K\}, \quad \Sigma_K := \left\{ \mu : v \rightarrow \int_{e_i} v d\sigma, i = 1, \dots, 4 \right\}.$$

In the following we use the functional

$$l_K(f) := \sum_{i=1}^4 \frac{(-1)^i}{\text{meas}(e_i)} \int_{e_i} f d\sigma. \quad (2.1)$$

It is easy to show that the functional  $l_K$  defines a linear functional on  $C^0(K)$ , and satisfies

$$l_K(p) = 0, \quad p \in P_1(K). \quad (2.2)$$

It is also worth noting that when  $q$  belongs to  $R_K$ , then  $q = p_q + \alpha_q B_K$ , with  $p_q \in P_1(K)$  and  $\alpha_q \in \mathbb{R}$ ; so, using (2.2), we get for all  $q \in R_K$ :

$$l_K(q) := \alpha_q l_K(B_K). \quad (2.3)$$

The next characterization result is the starting point of our nonconforming finite element.

**Theorem 1.** *Let  $l$  be the functional given by (2.3). Then the triple  $(K, R_K, \Sigma_K)$  is a finite element if and only if*

$$l_K(B_K) \neq 0. \quad (2.4)$$

**Proof.** *Necessity:* Let us assume to the contrary that  $l_K(B_K) = 0$ . Then, using the fact  $l_K(p) = 0$ , for all  $p$  affine function, we deduce from (2.3) that  $l_K(q) = 0$ , for all  $q \in R_K$ . Thus, there are no functions  $q_j \in R_K$  satisfying  $\int_{e_i} q_j d\sigma = \delta_i^j$ ,  $i = 1, \dots, 4$ . Consequently, the triple  $(K, R_K, \Sigma_K)$  is not a finite element. Hence, the necessary condition (2.4) is satisfied.

*Sufficiency:* Let us assume that condition (2.4) is satisfied. Then, using (2.2), we deduce that  $B_K \notin P_1(K)$  and then  $\dim(R_K) = 4 = \text{Card}(\Sigma_K)$ . Let  $q \in R_K$  such that

$$\int_{e_i} q d\sigma = 0, \quad i = 1, \dots, 4. \quad (2.5)$$

Now using (2.2), it is easy to see that  $0 = l_K(q) = \alpha l_K(B_K)$ , then  $\alpha = 0$ , and hence  $q \in P_1(K)$ . Thus, by using (2.5), the fact that normal derivatives of affine functions are constant and Green’s formula we get:

$$\int_K |\nabla q|^2 dx = - \int_K q \Delta q dx + \sum_{i=1}^4 \int_{e_i} q \frac{\partial q}{\partial \nu_i} d\sigma = \sum_{i=1}^4 \frac{\partial q}{\partial \nu_i} \int_{e_i} q d\sigma = 0.$$

This implies that  $q$  is a constant function on  $K$ . Finally, since  $\int_{e_1} q \, d\sigma = 0$ , we deduce that  $q \equiv 0$  on  $K$  and then  $R_K$  is  $\Sigma_K$ -unisolvant. This completes the proof of [Theorem 1](#).  $\square$

Let us note here that there exists a large class of functions satisfying [\(2.4\)](#).

**Example 1.** Let  $F_K$  be the invertible bilinear mapping that maps  $\hat{K} = [-1, 1] \times [-1, 1]$  onto  $K$  with  $a_i = F_K(\hat{a}_i)$ ,  $1 \leq i \leq 4$ , where  $\hat{a}_1 = (-1, -1)$ ,  $\hat{a}_2 = (1, -1)$ ,  $\hat{a}_3 = (1, 1)$ , and  $\hat{a}_4 = (-1, 1)$  are the vertices of  $\hat{K}$ . It is easy to verify that the function  $B_K = \frac{3}{8}(x^2 - y^2) \circ F_K^{-1}$  satisfies [\(2.4\)](#). Let us observe that in this particular case, we find exactly the rotated  $Q_1$  element proposed by Rannacher and Turek [\[11\]](#).

**Example 2.** Let  $i \in \{1, \dots, 4\}$  and the unique  $l_i \in P_3(K)$  satisfying  $l_i = 0$  on  $e_j$ ,  $i \neq j$  and  $\frac{1}{\text{meas}(e_i)} \int_{e_i} l_i \, d\sigma = 1$ . Then  $B_K = \sum_{i=1}^4 \frac{(-1)^i}{4} l_i$  satisfies [\(2.4\)](#).

**Example 3.** Let  $i \in \{1, \dots, 4\}$  and  $T_1, T_2$  be two adjacent triangles such that  $\bar{K} = \overline{\bigcup_{i=1}^2 T_i}$  and  $T_1 \cap T_2 = \emptyset$ . We consider the function  $B_K \in C^0$  defined by  $B_K|_{T_j} \in P_2(T_j)$ ,  $j = 1, 2$ ,  $B_K = 0$  on  $\partial T_1 \cap \partial T_2$  and for all edges  $e_i$ ,  $i = 1, \dots, 4$ ,  $\frac{1}{\text{meas}(e_i)} \int_{e_i} B_K \, d\sigma = (-1)^i$ . Then  $B_K$  satisfies [\(2.4\)](#).

### 3. A priori error estimate for nonconforming element

Let  $\Omega$  be a two-dimensional bounded open set with polygonal boundary  $\partial\Omega$ , and  $\mathcal{T}_h$  be a quadrangulation of  $\bar{\Omega}$  made by non-degenerate convex quadrilaterals with diameter bounded by  $h$ . Let  $K$  be one of these quadrilaterals, let  $\{a_i\}_{1 \leq i \leq 4}$  be its vertices numbered counterclockwise,  $\{e_i\}$  its edges  $[a_i, a_{i+1}]$  and  $S_i$  its sub-triangle with vertices  $a_{i-1}, a_i, a_{i+1}$  the indices being numbered modulo four. Let  $h_i$  be the diameter of  $S_i$ , and  $\rho_i$  the diameter of its inscribed circle. We shall write  $h_K = \sup_{1 \leq i \leq 4} h_i$ ,  $\rho_K = 2 \inf_{1 \leq i \leq 4} \rho_i$  and  $\sigma_K = \frac{h_K}{\rho_K}$ . Clearly,  $h_K$  is the diameter of  $K$ , and  $\sigma_K$  is a measure of the non-degeneracy of  $K$ . Here we also assume that the family of quadrangulations  $\mathcal{T}_h$  is regular, i.e., there exists a constant  $\sigma$ , independent of  $h$ , such that  $\sigma_K \leq \sigma$ , for all  $K \in \mathcal{T}_h$ . For each  $K \in \mathcal{T}_h$ , in the following analysis, the essential conditions on the function  $B_K$  will be:

$$l_K(B_K) := \sum_{i=1}^4 \frac{(-1)^i}{\text{meas}(e_i)} \int_{e_i} B_K \, d\sigma = 1, \quad |B_K|_{1,K} \leq C.$$

An immediate consequence of [Theorem 1](#) is that for each  $v \in H^1(K)$ , there exists a unique  $I_K(v) \in R_K$  such that

$$\int_{e_i} (I_K(v) - v) \, d\sigma = 0, \quad i = 1, \dots, 4. \tag{3.6}$$

**Theorem 2.** *There exists a constant  $C > 0$ , independent of  $v$  such that the operator  $I_K$  satisfies:*

$$|I_K(v)|_{1,K} \leq C \sum_{i=1}^4 \frac{1}{\text{meas}(e_i)} \left| \int_{e_i} v \, d\sigma \right|, \quad v \in H^1(K). \tag{3.7}$$

**Proof.** Here, we use  $C$  to represent a generic positive constant that can take different values at different occurrences. For the sake of convenience, we shall write  $\xi$  for  $\sum_{i=1}^4 \frac{1}{\text{meas}(e_i)} \left| \int_{e_i} v \, d\sigma \right|$ . Fix  $v \in H^1(K)$ . Since  $I_K(v)$  belongs to  $R_K$ , then  $I_K(v) = p + \alpha B_K$ , with  $p \in P_1(K)$  and  $\alpha \in \mathbb{R}$ . Consequently, we have  $l_K(v) = l_K(I_K(v)) = \alpha l_K(B_K) = \alpha$ , where  $l_K$  is the functional defined in [\(2.1\)](#). Hence, using the definition of  $l_K$ , we conclude that:

$$|\alpha| \leq \xi. \tag{3.8}$$

From the fact that  $\nabla I_K(v) = \nabla p + \alpha \nabla B_K$ , it follows by the triangle inequality that  $|\nabla I_K(v)|_{1,K} \leq |\nabla p|_{1,K} + |\alpha| |\nabla B_K|_{1,K}$ . Therefore, taking into account Eq. [\(3.8\)](#), it is easy to see that [\(3.7\)](#) holds if we show that  $|\nabla p|_{1,K} \leq C\xi$ , for some positive constant  $C$ . To this end, by Green's formula, we observe first that

$$\begin{aligned} \int_K |\nabla p|^2 \, dx &= \int_K \nabla I_K(v) \cdot \nabla p \, dx - \alpha \int_K \nabla B_K \cdot \nabla p \, dx \\ &= \sum_{i=1}^4 \int_{e_i} I_K(v) \frac{\partial p}{\partial \nu_i} \, d\sigma - \alpha \int_K \nabla B_K \cdot \nabla p \, dx, \end{aligned} \tag{3.9}$$

and since  $p \in P_1(K)$ ,  $\frac{\partial p}{\partial v_i} \in P_0(e_i)$ , then we get:

$$\left| \sum_{i=1}^4 \int_{e_i} I_K(v) \frac{\partial p}{\partial v_i} d\sigma \right| = \left| \sum_{i=1}^4 \int_{e_i} v \frac{\partial p}{\partial v_i} d\sigma \right| \leq h_K |\nabla p| \xi \leq C |p|_{1,K} \xi. \quad (3.10)$$

Also, by the Cauchy-Schwarz inequality, we have  $|\int_K \nabla B_K \cdot \nabla p dx| \leq C |p|_{1,K}$ . Thus, the desired result can be easily derived directly from (3.9) together with (3.8). This validates the bound given in (3.7).  $\square$

**Theorem 3.** The operator  $I_K$  satisfies, for all  $v \in H^{1+\varepsilon}(K)$  and  $0 \leq \varepsilon \leq 1$ ,

$$|v - I_K(v)|_{1,K} + h_K^{-1} \|v - I_K(v)\|_{0,K} \leq Ch_K^\varepsilon |v|_{1+\varepsilon,K}. \quad (3.11)$$

**Proof.** Let  $v \in H^{1+\varepsilon}(K)$  with  $0 \leq \varepsilon \leq 1$ . First, using Theorem 2 and the fact that the operator  $I_K$  preserves constants we have, for all  $\alpha \in \mathbb{R}$ ,

$$|I_K(v)|_{1,K} = |I_K(v - \alpha)|_{1,K} \leq C \sum_{i=1}^4 \frac{1}{\text{meas}(e_i)} \left| \int_{e_i} (v - \alpha) d\sigma \right|.$$

From the classical trace inequality [1], we get, for any  $u \in H^1(K)$ :

$$\frac{1}{\text{meas}(e_i)} \left| \int_{e_i} u d\sigma \right| \leq C \{ |u|_{1,K} + h_K^{-1} \|u\|_{0,K} \}, \quad i = 1, \dots, 4,$$

and then we have:

$$|I_K(v)|_{1,K} \leq C \inf_{\alpha \in \mathbb{R}} \{ |v|_{1,K} + h_K^{-1} \|v - \alpha\|_{0,K} \} \leq C |v|_{1,K}.$$

Now, let  $p \in P_1(K)$ , since  $I_K(p) = p$ , we have:

$$|v - I_K(v)|_{1,K} = |v - p - I_K(v - p)|_{1,K} \leq |v - p|_{1,K} + |I_K(v - p)|_{1,K} \leq C |v - p|_{1,K},$$

this implies:

$$|v - I_K(v)|_{1,K} \leq C \inf_{p \in P_1(K)} |v - p|_{1,K} \leq Ch_K^\varepsilon |v|_{1+\varepsilon,K}. \quad (3.12)$$

Finally, from (3.6), we can derive:

$$\|v - I_K(v)\|_{0,K} \leq Ch_K |v - I_K(v)|_{1,K} \leq Ch_K^{1+\varepsilon} |v|_{1+\varepsilon,K}. \quad (3.13)$$

Now, a combination of (3.12) and (3.13) yields the desired result.  $\square$

#### 4. Some considerations for the three-dimensional case

We only give a brief remark to extend the results of Sections 2 and 3 in three dimensions. Let  $K$  be a three-dimensional hexahedron, with  $F_j$ ,  $j = 1, \dots, 6$ , being the six faces such that  $F_j$  and  $F_k$  are opposite faces if  $j + k = 7$ . For all  $v \in H^1(K)$ , we set:

$$\begin{aligned} \mu_1(v) &= \frac{1}{\text{meas}(F_1)} \int_{F_1} v d\sigma + \frac{1}{\text{meas}(F_6)} \int_{F_6} v d\sigma \\ &\quad - \frac{1}{\text{meas}(F_2)} \int_{F_2} v d\sigma - \frac{1}{\text{meas}(F_5)} \int_{F_5} v d\sigma, \\ \mu_2(v) &= \frac{1}{\text{meas}(F_3)} \int_{F_3} v d\sigma + \frac{1}{\text{meas}(F_4)} \int_{F_4} v d\sigma \\ &\quad - \frac{1}{\text{meas}(F_2)} \int_{F_2} v d\sigma - \frac{1}{\text{meas}(F_5)} \int_{F_5} v d\sigma. \end{aligned}$$

Analogously as the two-dimensional case, we have

$$\mu_i(p) = 0, \quad p \in P_1(K), \quad i = 1, 2.$$

We consider two functions  $B_{K,1}, B_{K,2}$  in  $C^0(K)$  such that  $\mu_j(B_{K,i}) = \delta_i^j$ ,  $i, j = 1, 2$ . If we set

$$G_K = P_1(K) + \text{span}\{B_{K,1}, B_{K,2}\}, \quad \Sigma_K := \left\{ \mu : v \rightarrow \int_{F_i} v \, d\sigma, \quad i = 1, \dots, 6 \right\},$$

then we have the two following results.

**Theorem 4.** *The triple  $(K, G_K, \Sigma_K)$  is finite element.*

By [Theorem 4](#) there exists a unique  $I_K(v) \in G_K$  such that

$$\int_{F_i} (I_K(v) - v) \, d\sigma = 0, \quad i = 1, \dots, 6. \quad (4.14)$$

**Theorem 5.** *The operator  $I_K$  satisfies, for all  $v \in H^{1+\varepsilon}(K)$ ,  $0 \leq \varepsilon \leq 1$*

$$\|v - I_K(v)\|_{1,K} + h_K^{-1} \|v - I_K(v)\|_{0,K} \leq Ch_K^\varepsilon |v|_{1+\varepsilon,K}. \quad (4.15)$$

The proof of [Theorems 4 and 5](#) are similar to [Theorems 2 and 3](#).

## Acknowledgements

Authors thank the Volubilis Hubert-Curien Program (MA/13/286), LIRIMA, LIEM2I and Hassan-I University Fund for financial support. The authors thank Professor Allal Guessab for helpful discussion during the preparation of this manuscript.

## References

- [1] A. Agouzal, A posteriori error estimators for nonconforming approximation, *Int. J. Numer. Anal. Model.* 5 (1) (2008) 77–85.
- [2] T. Arbogast, Zhangxin Chen, On the implementation of mixed methods as nonconforming methods for second-order elliptic problems, *Math. Comput.* 64 (211) (1995) 943–972.
- [3] D.N. Arnold, D. Boffi, R.S. Falk, Approximation by quadrilateral finite elements, *Math. Comput.* 71 (2002) 909–922.
- [4] M. Crouzeix, P.A. Raviart, Conforming and nonconforming finite element methods for solving the stationary Stokes equations. I, *ESAIM Math. Model. Numer. Anal.* 7 (R3) (1973) 33–75.
- [5] J. Douglas Jr., J.E. Santos, D. Sheen, X. Ye, Nonconforming Galerkin methods based on quadrilateral elements for second order elliptic problems, *RAIRO Math. Model. Anal. Numer.* 33 (1999) 747–770.
- [6] H. Han, Nonconforming elements in the mixed finite element method, *J. Comput. Math.* 2 (3) (1984) 223–233.
- [7] H. Han, A finite element approximation of Navier–Stokes equations using nonconforming elements, *J. Comput. Math.* 2 (1) (1984) 77–88.
- [8] V. Girault, P.-A. Raviart, *Finite Element Methods for Navier–Stokes Equations*, Springer Verlag, 1986.
- [9] Jun Hu, Zhong-Ci Shi, Constrained quadrilateral nonconforming rotated  $Q_1$ -element, *J. Comput. Math.* 23 (2005) 561–586.
- [10] C. Park, D. Sheen,  $P_1$ -nonconforming quadrilateral finite element methods for second-order elliptic problems, *SIAM J. Numer. Anal.* 41 (2003) 624–640.
- [11] R. Rannacher, S. Turek, Simple nonconforming quadrilateral Stokes element, *Numer. Methods Partial Differ. Equ.* 8 (2) (1992) 97–111.
- [12] Zhou Gang, A class of arbitrarily convex quadrilateral elements for solving Navier–Stokes equations by nonconforming elements, *Math. Numer. Sin.* 8 (3) (1986) 258–274 (in Chinese).