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The six Grothendieck operations on o-minimal sheaves ☆

Les six opérations de Grothendieck sur les faisceaux o-minimaux

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ABSTRACT

In this note, we report on our work on the formalism of the Grothendieck six operations on o-minimal sheaves. As an application to the theory of definable groups, we see that the cohomology of a definably compact group with coefficients in a field is a connected, bounded, Hopf algebra of finite type.

RÉSUMÉ

Dans cette note, nous esquissons notre travail sur le formalisme des six opérations de Grothendieck sur les faisceaux o-minimaux. En tant qu'application à la théorie des groupes définissables, nous montrons que la cohomologie d'un groupe définissablement compact avec coefficients dans un corps est une algèbre de Hopf connexe, bornée, de type fini.

1. Complete supports on definable spaces

Let $\mathbb{M} = (M, <, (c)_{c\in\mathcal{C}}, (f)_{f\in\mathcal{F}}, (R)_{R\in\mathcal{R}})$ be a fixed o-minimal structure with definable Skolem functions. In the rest of the note we shall work in the *category of definable spaces with definable continuous maps* (cf. [4]), which we denote by Def.

A definable space X is *definably normal* if for every disjoint closed definable subsets Z_1 and Z_2 of X, there are disjoint open definable subsets U_1 and U_2 of X such that $Z_i \subseteq U_i$ for $i = 1, 2$.

Let X be a definable space and $K \subseteq X$ a definable subset. We say that K is *definably compact* (cf. [13]) if every definable curve $\alpha : (a, b) \to K$ is completable in K (i.e. limits in a^+ and b^- exist in K). With this definition, we have that a definable set $X \subseteq M^n$ with its induced topology is definably compact if and only if it is closed and bounded in M^n. A definable space is *definably completable* if it can be definably immersed as an open dense subset of a definably compact space.

A continuous definable map $f : X \to Y$ between definable spaces X and Y is called *definably proper* if for every definably compact definable subset K of Y, its inverse image $f^{-1}(K)$ is a definably compact definable subset of X. If we assume that

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the spaces X and Y are Hausdorff and locally definably compact (i.e. every point has a definably compact neighborhood), then $f : X \to Y$ is definably proper if and only if $f : X \to Y$ is universally closed and separated in the category Def.

The category $\widetilde{\mathrm{Def}}$ is the category whose objects are of the form $\tilde{X}$, the *o-minimal spectrum* of X (cf. [14]), where X is an object of Def and the morphisms are of the form $\tilde{f} : \tilde{X} \to \tilde{Y}$, the o-minimal spectrum of a morphism $f : X \to Y$ of Def. If Z is a definable subset of X, then $\tilde{Z}$ is said to be *constructible*. The functor just defined $\mathrm{Def} \to \widetilde{\mathrm{Def}}$ is an isomorphism of categories, so under the assumptions mentioned above we have: (i) $\tilde{f} : \tilde{X} \to \tilde{Y}$ is *proper* in $\widetilde{\mathrm{Def}}$ (i.e. universally closed and separated in the category $\widetilde{\mathrm{Def}}$) if and only if $f : X \to Y$ is definably proper; (ii) $\tilde{X}$ is *complete* (i.e. the projection $\tilde{X} \to \{\mathrm{pt}\}$ is proper in $\widetilde{\mathrm{Def}}$) if and only if X is definably compact.

As it is well known, a model $\mathbb{S}$ of the first-order theory of $\mathbb{M}$ over M determines a functor $\mathrm{Def} \to \mathrm{Def}(\mathbb{S})$ sending a definable space X to the $\mathbb{S}$-definable space $X(\mathbb{S})$ and a continuous definable map $f : X \to Y$ to the continuous $\mathbb{S}$-definable map $f^{\mathbb{S}} : X(\mathbb{S}) \to Y(\mathbb{S})$. Let $\tilde{f} : \tilde{X} \to \tilde{Y}$ be a morphism in $\widetilde{\mathrm{Def}}$, $\alpha \in \tilde{Y}$, $a \models \alpha$ a realization of α and $\mathbb{S}$ a prime model of the first-order theory of $\mathbb{M}$ over $\{a\} \cup M$. For each object $\tilde{X}$ of $\widetilde{\mathrm{Def}}$, we have a restriction map $r : \widetilde{X(\mathbb{S})} \to \tilde{X}$. A fundamental result of [1] states that the restriction r induces a homeomorphism $\widetilde{(f^{\mathbb{S}})^{-1}(a)} \to \tilde{f}^{-1}(\alpha)$. This allows us to make the following definition.

Definition 1.1. The *family of complete supports on* $\widetilde{(f^{\mathbb{S}})^{-1}}(a)$, denoted c, is the family of all closed subsets A with $A \subseteq \tilde{Z}$ for some closed complete constructible subset $\tilde{Z}$ of $\widetilde{(f^{\mathbb{S}})^{-1}}(a)$.

The *family of complete supports on* $\tilde{f}^{-1}(\alpha)$, still denoted c, is the inverse image by r of the family of complete supports on $\widetilde{(f^{\mathbb{S}})^{-1}}(a)$.

If X is definably completable by a definably normal definable space, then the family c becomes filtrant (i.e. for very $C \in c$ and every neighborhood V of C there is a neighborhood U of C in V with $\overline{U} \subseteq V$ and $\overline{U} \in c$).

2. Proper direct image

Let A be a commutative ring with finite weak global dimension and denote by $\mathrm{Mod}(A_X)$ the category of sheaves of A-modules on a topological space X. (We refer to [2,9] for classical sheaf theory.)

Below we will work in the category $\widetilde{\mathrm{Def}}$ and omit the tilde for simplicity. Let $f : X \to Y$ be a morphism in Def and let $F \in \mathrm{Mod}(A_X)$. We note that by [5], $\mathrm{Mod}(A_X)$ is equivalent to the category of sheaves on the o-minimal site on the definable space corresponding to X, and o-minimal sites generalize the semi-algebraic site on semi-algebraic spaces (cf. [3]) and the sub-analytic site on globally sub-analytic sets (cf. [10,15]).

Definition 2.1. The *proper direct image* is the subsheaf of $f_* F$ defined by setting for every open constructible subset U of Y

$$\Gamma(U; f_{\wr} F) = \varinjlim_{Z} \Gamma_Z\big(f^{-1}(U); F\big),$$

where Z ranges through the family of closed constructible subsets of $f^{-1}(U)$ such that $f_{|Z} : Z \to U$ is proper.

The functor $f_{\wr}$ is well defined, stable under composition and left exact. If $f : X \to Y$ is proper, then $f_{\wr} = f_*$; if $f : X \to Y$ is the inclusion of a locally closed subset, then $f_{\wr}$ is the extension by zero functor; if we consider the morphism $a_X : X \to \{\mathrm{pt}\}$, then we have $a_{X\wr} F \simeq \Gamma_c(X; F)$ (sections with complete support).

In order to compute the fiber formula, we shall work in (the tilde of) a full subcategory **A** of the category Def such that:

(A0) every locally closed definable subset of an object of **A** is an object of **A**;
(A1) every object of **A** is definably normal (in fact completely definably normal);
(A2) every object of **A** is definably completable in **A**.

We have the following fiber formula: let $\alpha \in Y$. Then

$$(f_{\wr} F)_\alpha \simeq \Gamma_c\big(f^{-1}(\alpha); F\big) \simeq \Gamma_c\big(\big(f^{\mathbb{S}}\big)^{-1}(a); r^{-1} F\big). \tag{1}$$

Consider a fiber $f^{-1}(\alpha)$ of a morphism $f : X \to Y$ in Def and let c be the family of complete supports on $f^{-1}(\alpha)$. A sheaf F on $f^{-1}(\alpha)$ is *c-soft* if and only if the restriction $\Gamma(f^{-1}(\alpha); F) \to \Gamma(K; F)$ is surjective for every $K \in c$.

Definition 2.2. Let $f : X \to Y$ be a morphism in Def and let F be a sheaf in $\mathrm{Mod}(A_X)$. We say that F is *f-soft* if for any $\alpha \in Y$ its restriction $F_{|f^{-1}(\alpha)}$ is c-soft in $\mathrm{Mod}(A_{f^{-1}(\alpha)})$.

Let $F \in \mathrm{Mod}(A_X)$. Using the fiber formula (1) and the work in [7] we obtain the following properties:

$$\begin{cases} f\text{-soft sheaves are cogenerating;} \\ f_{\wr} \text{ has finite cohomological dimension (in our case } R^k f_{\wr} F = 0 \text{ if } k > \dim X); \\ f\text{-soft sheaves are } f_{\wr}(\bullet)\text{-acyclic (i.e. } R^k f_{\wr} F = 0,\ k \neq 0 \text{ if } F \text{ is } f\text{-soft);} \\ f\text{-soft sheaves are stable under small } \oplus; \\ f_{\wr} \text{ commutes with small } \oplus. \end{cases} \tag{2}$$

3. Fundamental formulas

For some of our results about the proper direct image (base change formula, Künneth formula), we will also require that:

(A3) $f : X \to Y$ is a morphism in Def and if $u \in Y$, then for every elementary extension $\mathbb{S}$ of $\mathbb{M}$ and every $F \in \mathrm{Mod}(A_{X_{\mathrm{def}}})$, we have an isomorphism

$$H_c^*\big(\tilde{f}^{-1}(u); \tilde{F}_{|\tilde{f}^{-1}(u)}\big) \simeq H_c^*\big(\widetilde{(f^{\mathbb{S}})}^{-1}(u); \tilde{F}(\mathbb{S})_{|\widetilde{(f^{\mathbb{S}})}^{-1}(u)}\big),$$

where $\tilde{F}(\mathbb{S}) = r^{-1}\tilde{F}$ and $r : \widetilde{X(\mathbb{S})} \to \tilde{X}$ is the restriction.

Categories **A** satisfying also (A3) include: (i) regular, locally definably compact definable spaces in o-minimal expansions of real closed fields; (ii) Hausdorff locally definably compact definable spaces in o-minimal expansions of ordered groups with definably normal completions; (iii) locally closed definable subspaces of Cartesian products of a given definably compact definable group in an arbitrary o-minimal structure (for this case we have a weaker version of (A1) which is enough for the applications).

Theorem 1 *(Projection formula). Let $f : X \to Y$ be a morphism in* **A***. Let $F \in \mathrm{D}^+(A_X)$ and $G \in \mathrm{D}^+(A_Y)$. Then there is a natural isomorphism:*

$$Rf_{\wr} F \otimes G \simeq Rf_{\wr}(F \otimes f^{-1}G).$$

Consider a Cartesian square in **A** where $\delta = f \circ g' = g \circ f'$

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y \\ {\scriptstyle g'}\downarrow & \searrow^{\delta} & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y' \end{array}$$

Theorem 2 *(Base change formula). Suppose that $f : X \to Y$ satisfies (A3). Then there is a natural isomorphism in $\mathrm{D}^+(A_Y)$, functorial in $F \in \mathrm{D}^+(A_X)$:*

$$g^{-1} \circ Rf_{\wr} F \simeq Rf'_{\wr} \circ g'^{-1} F.$$

Theorem 3 *(Künneth formula). Suppose that $f : X \to Y$ satisfies (A3). Then For $F \in \mathrm{D}^+(A_X)$ and $G \in \mathrm{D}^+(A_{Y'})$ there is a natural isomorphism:*

$$R\delta_{\wr}\big(g'^{-1}F \otimes f'^{-1}G\big) \simeq Rf_{\wr} F \otimes Rg_{\wr} G.$$

When $Y' = \mathrm{pt}$, $X' = X \times Y$ and the maps f, f', g, g' are the projections, as a special case of the Künneth formula we obtain:

$$H_c^k(X \times Y; A_{X\times Y}) \simeq \bigoplus_{p+q=k} \big(H_c^q(X; A_X) \otimes H_c^p(Y; A_Y)\big), \quad k \in \mathbb{Z}.$$

Let $f : X \to Y$ be a morphism in **A**. As a consequence of (2) and the Brown representability (cf. [11])

Theorem 4 *(Verdier duality). The derived functor $f^{\wr} : \mathrm{D}^+(A_Y) \to \mathrm{D}^+(A_X)$ is well defined and it is the right adjoint to $Rf_{\wr} : \mathrm{D}^+(A_X) \to \mathrm{D}^+(A_Y)$.*

In particular, we obtain the global Poincaré–Verdier duality (cf. [7]). Here $\overset{\wr}{a_X^{\cdot}} A_X$ is the *dualizing complex* and F varies through $\mathrm{D}^b(A_X)$. There is a natural isomorphism:

$$R\operatorname{Hom}\big(F, \overset{\wr}{a_X^{\cdot}} A\big) \simeq R\operatorname{Hom}\big(R\varGamma_c(X;F), A\big).$$

4. Application to definable groups

Let $\mathbb{M}$ be an arbitrary o-minimal structure and k a field. Let X be an object of Def. A result of classical sheaf theory (cf. [2]) states that there is a cup product operation

$$\cup : H^p(X;k_X) \otimes H^q(X;k_X) \to H^{p+q}(X;k_X)$$

making $H^*(X;k_X)$ into a graded, associative, skew-commutative k-algebra with unit in $H^0(X;k_X)$. This product is functorial and the algebra is connected if X is definably connected. In combination with the cohomological results from [5,8], just like in [6], we also find the following application to the theory of definable groups (cf. [12]).

Theorem 5. *Suppose that $\mathbb{M}$ is an arbitrary o-minimal structure. Let k be a field. If G is a definably connected, definably compact definable group, then the o-minimal sheaf cohomology $H^*(G;k_G)$ of G with coefficients in k is a connected, bounded, Hopf algebra over k of finite type.*

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