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Sharp L^p estimates for discrete second-order Riesz transforms

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ABSTRACT

We show that multipliers of second-order Riesz transforms on products of discrete Abelian groups enjoy the L^p estimate (p^*-1), where $p^*=\max\{p,q\}$, $1/p+1/q=1$. This estimate is sharp for certain multipliers such as $R_1R_1^*-R_2R_2^*$ on products of infinite groups. For other multipliers such as $R_1R_1^*$, the best possible estimate is given by the Choi constant. Those are the first known sharp L^p estimates of discrete Calderón–Zygmund operators.

RÉSUMÉ

Nous montrons que les carrés des transformations de Riesz sur des produits de groupes abéliens discrets ont une norme L^p bornée par la constante (p^*-1), avec $p^*=\max\{p,q\}$, $1/p+1/q=1$. Cette constante est optimale dans le cas de goupes infinis pour certains opérateurs, parmi lesquels $R_1R_1^*-R_2R_2^*$. Pour d'autres opérateurs, parmi lesquels $R_1R_1^*$, la constante optimale est donnée par la constante de Choi. Il s'agit des premières estimations L^p optimales connues d'opérateurs discrets de type Calderón–Zygmund.

Version française abrégée

Nous nous intéressons à des opérateurs discrets de type Calderón–Zygmund sur des groupes abéliens tels que $\mathbb{Z}$ et $\mathbb{Z}/m\mathbb{Z}$, ainsi qu'à leurs produits. Les transformées discrètes de Riesz ou de Hilbert sont des exemples de tels opérateurs. Ils sont définis à l'aide de dérivées discrètes, ce qui induit de nombreuses difficultés par rapport au cas continu du fait du caractère non local de ces opérateurs de dérivation. L'estimation optimale des opérateurs de Riesz et Hilbert discrets est toujours un problème ouvert. Il est donc remarquable que les carrés des transformations de Riesz discrètes puissent être estimés par les mêmes constantes que dans le cas continu [7].

Pour illustrer la particularité du cas discret, nous pouvons citer les travaux de F. Lust-Piquard, qui montre dans [3] que la transformée de la fonction carrée du vecteur de Riesz discret sur les produits G^N de groupes abéliens possède une norme L^p indépendante de la dimension N lorsque $p \geqslant 2$. Elle utilise pour cela des méthodes non commutatives. Elle fournit par ailleurs un exemple montrant la dépendance en la dimension N lorsque $p<2$. Cette dépendance dimensionnelle pour certains p dans le cas discret s'oppose donc fortement au cas continu, pour lequel de nombreux résultats indépendants de la dimension existent. Nous renvoyons à des résultats classiques de Stein [6]. Des méthodes probabilistes utilisées par P.-A. Meyer [4] ont donné la première preuve de ces résultats dans le cas gaussien, pour lesquels G. Pisier [5] a donné une preuve analytique.

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Soit donc $G^N := \bigotimes_{i=1}^N G_i$, $G_i \in \{\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}; m \geqslant 2\}$ un produit de groupes additifs abéliens de générateurs e_i et $f : G^N \to \mathbb{C}$. On définit de manière classique les i-èmes dérivations $\partial^i_\pm$ discrètes à droite et à gauche par $\partial^i_+ f(n) = f(n+e_i) - f(n)$ et $\partial^i_- f(n) = f(n) - f(n-e_i)$. Le laplacien discret est $\Delta = \sum_{i=1}^N \partial^i_+ \partial^i_-$. Les transformées de Riesz discrètes sont définies par $R^i_+ \circ \sqrt{-\Delta} = \partial^i_+$ et $R^i_- \circ \sqrt{-\Delta} = \partial^i_-$. On note R^2_α avec $\alpha \in \mathbb{C}^N$ une combinaison linéaire des carrés des transformations de Riesz comme défini en (1).

Nos deux principaux résultats sont les estimations optimales de R^2_α dans L^p énoncées plus bas dans les Théorèmes 1.1 et 1.2.

Les preuves de ces résultats utilisent la technique déterministe des fonctions de Bellman. Pour le Théorème 1.1, on construit une fonction de Bellman associée à l'estimation optimale recherchée. L'opérateur R^2_α est alors écrit sous la forme faible $(f, R^2_\alpha g)$ donnée dans (4) et faisant intervenir les extensions de la chaleur $\tilde{f}(t) := \mathrm{e}^{t\Delta} f$ et $\tilde{g}(t) := \mathrm{e}^{t\Delta} g$. Des estimations de dissipation (2) liées aux propriétés de concavité de la fonction de Bellman donnent accès à des estimations bilinéaires (3), puis au résultat voulu. La preuve du Théorème 1.2 suit la même stratégie et nécessite la construction d'une fonction de Bellman adaptée jouissant de propriétés de concavité originales.

On peut montrer que les estimations obtenues sont optimales pour des produits de groupes infinis.

1. Introduction

Sharp L^p estimates for norms of Calderón–Zygmund operators are a cornerstone in harmonic analysis, but such estimates are rare. They are known for Hilbert and Riesz transforms as well as second-order Riesz transforms. There are, however, no known results when entering discrete Abelian groups such as $\mathbb{Z}$ or $\mathbb{Z}/m\mathbb{Z}$. The discrete derivatives that come into play in order to define Riesz and Hilbert transforms induce difficulties that are due to their non-local nature. To underline the artifacts one may encounter, we briefly discuss a related subject, the dimensional behavior of Riesz vectors in a variety of different situations.

F. Lust-Piquard showed in [3], using non-commutative methods, that square functions of Riesz transforms on certain products of discrete Abelian groups of a single generator enjoy dimension-free L^p norms when $p \geqslant 2$. She provided an example demonstrating dimensional growth when $p < 2$. This difficulty of dimensional dependence for some p in the discrete case is in a sharp contrast to numerous dimension-free estimates in some very general continuous settings. We refer the reader to the result by Stein [6] in the classical case. Probabilistic methods, applied by P.-A. Meyer [4] lead to the first proof of this theorem in the Gaussian setting, whereas G. Pisier [5] found an analytic proof.

Let G be an additive discrete Abelian group generated by e and $f : G \to \mathbb{C}$. Its right- and left-hand derivatives are $\partial_+ f(n) = f(n+e) - f(n)$ and $\partial_- f(n) = f(n) - f(n-e)$. We then define partial derivatives $\partial^i_\pm$ accordingly on products $\bigotimes_{i=1}^N G_i$ of such groups. Here, we denote by e_i the ith unit element in G^N and $\partial^i_+ f(n) = f(n+e_i) - f(n)$ and $\partial^i_- f(n) = f(n) - f(n-e_i)$. For simplicity, we expose the case of a product of the form $G^N := \bigotimes_{i=1}^N G$, but it will be clear that our methods apply to mixed cases $\bigotimes_{i=1}^N G_i$ where G_i are not necessarily the same discrete groups. The discrete Laplacian becomes $\Delta = \sum_{i=1}^N \partial^i_+ \partial^i_-$.

For each direction i, there are two choices of Riesz transforms, defined in a standard manner: $R^i_+ \circ \sqrt{-\Delta} = \partial^i_+$ and $R^i_- \circ \sqrt{-\Delta} = \partial^i_-$. In this text, we are concerned with second-order Riesz transforms defined as $R^2_i := R^i_+ R^i_-$ and combinations thereof, such as

$$R^2_\alpha := \sum_{i=1}^N \alpha_i R^2_i, \qquad \alpha = \{\alpha_i\} \in \mathbb{C}^N : |\alpha_i| \leqslant 1. \tag{1}$$

By the use of Bellman functions, we prove the following.

Theorem 1.1. *$R^2_\alpha : L^p(G^N, \mathbb{C}) \to L^p(G^N, \mathbb{C})$ enjoys the operator norm estimate $\|R^2_\alpha\| \leqslant p^* - 1$. Here, $p^* - 1 = \max\{p-1, 1/(p-1)\}$. This estimate is sharp when G is infinite and $N \geqslant 2$.*

Better estimates are available in the real valued case:

Theorem 1.2. *$R^2_\alpha : L^p(G^N, \mathbb{R}) \to L^p(G^N, \mathbb{R})$ with $\alpha = \{\alpha_i\}_{i=1,\dots,N} \in \mathbb{R}^N$ enjoys the norm estimate $\|R^2_\alpha\| \leqslant \mathfrak{C}_{\min\alpha, \max\alpha, p}$, where these are the Choi constants (see [2] for their definition).*

The sharpness assertion of Theorems 1.1 and 1.2 follows from the sharpness of the constant in the continuous setting $G^N = \mathbb{R}^N$ by using approximation arguments.

2. Elements of the proof

The existence of the Bellman function $B_{\varepsilon,p,K}(\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{f}, \boldsymbol{g})$ we are interested in follows from Burkholder's theorem [1] on L^p norms of differentially subordinate martingales and has the following properties: for any $1 < p < \infty$, we define $D_p :=$

$\{\boldsymbol{v} := (\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{f}, \boldsymbol{g}) \subset \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{C} \times \mathbb{C} : |\boldsymbol{f}|^p < \boldsymbol{F}, |\boldsymbol{g}|^q < \boldsymbol{G}\}$. Let K be any compact subset of D_p and let ε be a sufficiently small positive number. Then there exists a function $B_{\varepsilon,p,K}(\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{f}, \boldsymbol{g})$ that is infinitely differentiable in an ε-neighborhood of K and such that for all $\boldsymbol{v} \in K$, $0 \leqslant B_{\varepsilon,p,K}(\boldsymbol{v}) \leqslant (1 + \varepsilon C_K)(p^* - 1)\boldsymbol{F}^{1/p}\boldsymbol{G}^{1/q}$ and $-\mathrm{d}^2_{\boldsymbol{v}} B_{\varepsilon,p,K}(\boldsymbol{v}) \geqslant 2|\mathrm{d}\boldsymbol{f}||\mathrm{d}\boldsymbol{g}|$ (see [7]).

We are going to use in a decisive way that the right-hand side of the last inequality above does not depend on $\boldsymbol{v}$. The latter is translated into dissipation estimates in the lemma below, where $\tilde{f}$ denotes the semi-continuous heat extension of a function f. Namely, $\tilde{f}(t) = e^{t\Delta} f$, where Δ is the discrete Laplacian.

Lemma 2.1. *Let us assume a compact and convex subset $K \subset D_p$ has been chosen. Let $\tilde{\boldsymbol{v}}(n,t) := (\widetilde{|f|^p}(n,t), \widetilde{|g|^q}(n,t), \tilde{f}(n,t), \tilde{g}(n,t))$ and assume that $\tilde{\boldsymbol{v}}(n,t)$ and its neighbors $\tilde{\boldsymbol{v}}(n \pm e_i, t)$ lie in the domain K of $B_{\varepsilon,p,K}$. Then*

$$(\partial_t - \Delta)(B_{\varepsilon,p,K} \circ \tilde{\boldsymbol{v}})(n,t) \geqslant \sum_i \big(|\delta^i_+ \tilde{f}||\delta^i_+ \tilde{g}| + |\delta^i_- \tilde{f}||\delta^i_- \tilde{g}|\big). \tag{2}$$

Proof. Omitting the smoothness parameters writing $B := B_{\varepsilon,p,K}$, we get by the use of Taylor's theorem:

$$(\partial_t - \Delta)(B \circ \tilde{\boldsymbol{v}}) = \nabla_{\boldsymbol{v}} B(\tilde{\boldsymbol{v}})(\partial_t - \Delta)\tilde{\boldsymbol{v}} + \sum_{i,\pm} \int_0^1 (1-s)\big(-\mathrm{d}^2_{\boldsymbol{v}} B\big(\tilde{\boldsymbol{v}} + s\delta^i_\pm \tilde{\boldsymbol{v}}\big)\delta^i_\pm \tilde{\boldsymbol{v}}, \delta^i_\pm \tilde{\boldsymbol{v}}\big)\mathrm{d}s.$$

First, observe that $(\partial_t - \Delta)\tilde{\boldsymbol{v}} = 0$. Notice also that the domain of B is convex. At (n,t) both $\tilde{\boldsymbol{v}}$ and $\tilde{\boldsymbol{v}} + \delta^i_\pm \tilde{\boldsymbol{v}} = \tilde{\boldsymbol{v}}(\cdot \pm e_i, \cdot)$ are contained in the domain of B, and so is $\tilde{\boldsymbol{v}} + s\delta^i_\pm \tilde{\boldsymbol{v}}$ for $0 \leqslant s \leqslant 1$. Together with the universal inequality $-\mathrm{d}^2_{\boldsymbol{v}} B(\boldsymbol{v}) \geqslant 2|\mathrm{d}\boldsymbol{f}||\mathrm{d}\boldsymbol{g}|$ in the domain of B, we obtain the desired estimate. □

Lemma 2.2. *Let f and g be test functions on G^N and p and q conjugate exponents. Then we have the bilinear embedding estimate*

$$2\sum_{i=1}^N \int_0^\infty \sum_{G^N} \big|\partial^i_+ \tilde{f}(n,t)\big|\big|\partial^i_+ \tilde{g}(n,t)\big|\mathrm{d}t \leqslant \big(p^* - 1\big)\|f\|_p \|g\|_q. \tag{3}$$

The proof is by an exhaustion argument that uses integration by parts and Green's identity formulae adapted to this discrete setting. The argument relies on the group structure of G.

This estimate is very useful to us, due to the following representation formula for a given second-order Riesz transform. It be seen through Fourier analysis or by the use of semigroups.

Lemma 2.3. *Let $i \in \{1, \ldots, N\}$. If $\hat{g}(0) = 0$ then the sums and integrals that arise below converge absolutely and:*

$$\big(f, R_i^2 g\big) = -2\int_0^\infty \sum_{G^N} \partial^i_+ \tilde{f}(n,t)\overline{\partial^i_+ \tilde{g}(n,t)}\mathrm{d}t. \tag{4}$$

Proof of Theorem 1.1. Remember that $R^2_\alpha := \sum_{i=1}^N \alpha_i R_i^2 = \sum_{i=1}^N \alpha_i R^i_+ R^i_-$, $\alpha_i \in \mathbb{C}$, $|\alpha_i| \leqslant 1$, $\forall i \in [1, \ldots, N]$. Using successively the representation formula (4) of Lemma 2.3 and the bilinear estimate (3) of Lemma 2.2, we have:

$$\begin{aligned}
\big|\big(f, R^2_\alpha g\big)\big| &:= \left|\sum_i \alpha_i \int_0^\infty \sum_{G^N} \partial^i_+ \tilde{f}(n,t)\overline{\partial^i_+ \tilde{g}(n,t)} + \partial^i_- \tilde{f}(n,t)\overline{\partial^i_- \tilde{g}(n,t)}\mathrm{d}t\right| \\
&\leqslant \sum_i |\alpha_i| \int_0^\infty \sum_{G^N} \big|\partial^i_+ \tilde{f}(n,t)\big|\big|\partial^i_+ \tilde{g}(n,t)\big| + \big|\partial^i_- \tilde{f}(n,t)\big|\big|\partial^i_- \tilde{g}(n,t)\big|\mathrm{d}t \\
&\leqslant \big(p^* - 1\big)\|f\|_p \|g\|_q. \qquad \square
\end{aligned}$$

We illustrate the necessary changes for the second result, involving the Choi constants.

Proof of Theorem 1.2. We present only the case $\alpha_i \in \{0, 1\}$. The strategy of proof relies again on the construction of two suitable Bellman functions $C^\pm_p$ derived from the result of Choi [2]: if T_σ is the martingale transform associated with the sequence $\sigma = (\sigma_0, \sigma_1, \ldots)$ with $\sigma_i \in \{0, 1\}$, then $\sup_\sigma \|T_\sigma\|_p \leqslant \mathfrak{C}_p$, for every $1 < p < \infty$. We derive the existence of functions

$C^{\pm}_{p,K,\varepsilon}$ of four variables $\boldsymbol{v} = (\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{f}, \boldsymbol{g})$ defined in an ε-neighborhood of a compact $K \subset D_p$, so that $0 \leqslant C^{\pm}_{\varepsilon,p,K}(\boldsymbol{v}) \leqslant (1 + \varepsilon C_K)\mathfrak{C}_p \boldsymbol{F}^{1/p}\boldsymbol{G}^{1/q}$ and so that we have the dissipation estimate $-\mathrm{d}^2 C^{\pm}_{\varepsilon,p,K}(\boldsymbol{v}) \geqslant \pm(1 \pm \operatorname{sign}(\mathrm{d}\boldsymbol{f}\,\mathrm{d}\boldsymbol{g}))\mathrm{d}\boldsymbol{f}\,\mathrm{d}\boldsymbol{g}$.

By the identity formula, we see that for $c_i \in \{0, 1\}$,

$$\left|\left(f, \sum_i c_i R_i^2 g\right)\right| = 2\left|\int_0^\infty \sum_{\mathbb{Z}^N} \sum_i c_i \partial^i_+ \tilde{f}(n,t)\partial^i_+ \tilde{g}(n,t)\mathrm{d}t\right|$$
$$\leqslant \max_{\pm}\left\{2\int_0^\infty \sum_{\mathbb{Z}^N}\sum_i \left[\partial^i_+ \tilde{f}\,\partial^i_+ \tilde{g}\right]_{\pm}\mathrm{d}t\right\}$$
$$\leqslant \mathfrak{C}_p \|f\|_p \|g\|_q.$$

Here $[\cdot]_{\pm}$ denotes positive and negative parts. We used successively the representation formula (4) the bilinear embedding resulting from the dissipation estimate for $C^{\pm}_p$ above. This concludes the proof of Theorem 1.2. □

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