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# Brin–Katok formula for the measure theoretic $r$-entropy

## *Formule de Brin–Katok pour la mesure de la r-entropie au sens de la théorie de la mesure*

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ABSTRACT

Entropy is undoubtedly among the most essential characteristics of dynamical systems. Calculations of various entropies are important but often difficult. This article is devoted to constructing the Brin–Katok formula for the measure theoretic $r$-entropy.


RÉSUMÉ

L'entropie constitue une notion éssentielle de la théorie des systèmes dynamiques. Les calculs des diverses entropies sont importants, mais souvent difficiles. On donne ici la formule structurelle de Brin–Katok pour la $r$-entropie au sens de la théorie de la mesure.


## 1. Introduction

A topological dynamical system is a triple $(X, d, T)$ (or tuple $(X, T)$ for short) consisting of a compact metric space $(X, d)$ and a continuous map $T : X \to X$. For $n \in \mathbb{N}$, the Bowen metric $d_n$ is given by $d_n(x, y) = \max\{d(T^i x, T^i y) : i = 0, 1, 2, \cdots, n-1\}$, for $x, y \in X$. Given $\epsilon > 0$, let $B_{d_n}(x, \epsilon) = \{y \in X : d_n(x, y) \le \epsilon\}$ be the $d_n$-ball about $x$ of radius $\epsilon$. We also write $B_n(x, \epsilon)$ for convenience, when there is no confusion. Suppose that $\mu$ is an ergodic measure on $X$, Brin and Katok [1] (see also [5]) proved that for $\mu$-almost every $x \in X$,

$$\lim_{\epsilon\to 0} \liminf_{n\to\infty} \frac{-\log \mu(B_n(x, \epsilon))}{n} = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{-\log \mu(B_n(x, \epsilon))}{n} = h_\mu(T), \tag{1.1}$$

where $h_\mu(T)$ is the measure theoretic entropy. Very recently, Zhu [13,14] proved the Brin–Katok formula above in the case of random dynamical systems. Recently, C. Pfister and W. Sullivan [6] defined the Bowen ball with a mistake function. More precisely,

**Definition 1.1.** Let $g : \mathbb{N} \to \mathbb{N}$ be a given non-decreasing unbounded map with properties:

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$$g(n) < n \quad \text{and} \quad \lim_{n\to\infty} \frac{g(n)}{n} = 0.$$

The function $g$ is called mistake function (blowup function there). Let $x \in X$ and $\epsilon > 0$, the mistake Bowen ball $B_n(g; x, \epsilon)$ is the closed set

$$B_n(g; x, \epsilon) := \left\{ y \in X : \exists \Lambda \subset \Lambda_n, \#(\Lambda_n \setminus \Lambda) \le g(n) \text{ and } \max\left\{ d\left(T^j x, T^j y\right) : j \in \Lambda \right\} \le \epsilon \right\},$$

where $\Lambda_n = \{0, 1, 2, \cdots, n-1\}$.

After that, the reader is referred to [2,4,8–12,15] for many interesting results about the mistake Bowen ball. Such results reveal that when a physical process evolves, it is natural that it may change or that some errors are committed in the evaluation of orbits and, if the system is self-adaptable, the proportion of errors should decrease as time evolves.

Recently, Y. Ren, L. He, J. Lu and G. Zheng [7] introduced the concept of measure theoretic $r$-entropy of a continuous map on a compact metric space, and obtained the result that measure theoretic entropy is the limit of measure theoretic $r$-entropy. It is worth pointing out that measure theoretic $r$-entropy in [7] is defined in view of Katok's topological version of measure theoretic entropy.

In this paper, inspired by the idea of Brin and Katok [1], we define the measure theoretic $r$-entropy in a local way. J. Ma and Z. Wen [3] showed that the Bowen entropy can be determined via the local entropies of measures, which can be considered as an analogue of Billingsley's Theorem for the Hausdorff dimension. And we show that the limit of measure theoretic $r$-entropy is equal to the measure theoretic entropy as $r \to 0$.

Now for $x \in X, n \ge 0, \epsilon > 0$ and $0 < r < 1$, let

$$B(x, n, \epsilon, r) := \left\{ y \in X : \frac{1}{n} \#\left\{ 0 \le i \le n-1 : d\left(T^i x, T^i y\right) \le \epsilon \right\} > 1 - r \right\}.$$

Obviously, $B(x, n, \epsilon, r) \supseteq B_n(x, \epsilon)$.

The main result of this article is as follows:

**Theorem 1.1.** *Let $(X, d, T)$ be a topological dynamical system. Suppose $\mu$ is an ergodic measure on $(X, T)$. Then for $\mu$-almost every $x \in X$, we have:*

$$\begin{aligned} h_\mu(T) &= \lim_{r\to 0} \lim_{\epsilon\to 0} \liminf_{n\to\infty} \frac{-\log \mu(B(x, n, \epsilon, r))}{n} \\ &= \lim_{r\to 0} \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{-\log \mu(B(x, n, \epsilon, r))}{n}. \end{aligned}$$

## 2. Proof of the main theorem

**Proof.** (1) Firstly, we want to show that for any $r > 0$, $\lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{-\log \mu(B(x,n,\epsilon,r))}{n} \le h_\mu(T)$. Since $B(x, n, \epsilon, r) \supseteq B_n(x, \epsilon)$ and formula (1.1), we have

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{-\log \mu(B(x, n, \epsilon, r))}{n} \le \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{-\log \mu(B_n(x, \epsilon))}{n} = h_\mu(T),$$

for $\mu$-almost all $x \in X$.

(2) Secondly, we will prove that $\lim_{r\to 0} \lim_{\epsilon\to 0} \liminf_{n\to\infty} \frac{-\log \mu(B(x,n,\epsilon,r))}{n} \ge h_\mu(T)$. Fix $\epsilon > 0$. There exists a finite measurable partition $\xi$ of $X$ satisfying:

- $h_\mu(T, \xi) \ge h_\mu(T) - \epsilon$,
- $\mu(\partial\xi) = 0$,

where $h_\mu(T, \xi)$ is the measure entropy with respect to $\xi$ and $\partial\xi$ denotes the boundary of the partition $\xi$. For $\theta > 0$, let

$$U_\theta(\xi) = \left\{ x \in X : \text{the ball } B(x, \theta) \text{ is not contained in } C_\xi(x) \right\},$$

where $C_\xi(x)$ denotes the element of the partition $\xi$ containing $x$. Since $\bigcap_{\theta>0} U_\theta(\xi) = \partial\xi$, we have that

$$\mu\left(U_\theta(\xi)\right) \to 0, \quad \text{as } \theta \to 0.$$

Therefore, there exists $\delta_0 > 0$ such that $\mu(U_\delta(\xi)) \le \epsilon$ for any $0 < \delta \le \delta_0$. Hence, by the Birkhoff ergodic theorem, for $\mu$-almost every $x \in X$ there exists $N_1(x) > 0$ such that for any $n \ge N_1(x)$,

$$\frac{1}{n} \sum_{i=0}^{n-1} \chi_{U_\delta(\xi)}\left(T^i(x)\right) \le \epsilon,$$

where $\chi_A$ is the characteristic function of the set $A$. Let $A_l := \{x \in X : N_1(x) \le l\}$. Clearly, the sets $A_l$ are nested and exhaust $X$ up to a set of measure zero. Therefore, fix $\gamma > 0$, there exists $l_0 > 1$ such that $\mu(A_l) \ge 1 - \gamma$ for any $l \ge l_0$.

Fix $l \ge l_0$. Given a point $x \in X$, we call the collection

$$\big(C_\xi(x), C_\xi\big(T(x)\big), \cdots, C_\xi\big(T^{n-1}(x)\big)\big)$$

the $(\xi, n)$-name of $x$. If $y \in B(x, \delta)$, then either $x$ and $y$ belong to the same element of $\xi$ or $x, y \in U_\delta(\xi)$. (The two cases may happen at the same time.) For $n \in \mathbb{N}$ and $\xi$, we give a metric $d_n^\xi$ between $(\xi, n)$-names of $x$ and $y$ as follows:

$$d_n^\xi\big((\xi, n)\text{-name of } x, (\xi, n)\text{-name of } y\big) = \frac{1}{n}\#\big\{0 \le i \le n-1 : C_\xi\big(T^i x\big) \ne C_\xi\big(T^i y\big)\big\}.$$

It can also be viewed as a semi-metric on $X$. Hence, if $x \in A_l, n > l$ and $y \in B(x, n, \delta, r)$, the distance $d_n^\xi$ between $(\xi, n)$-names of $x$ and $y$ does not exceed $r + \epsilon$. Furthermore, for $x \in A_l$, $B(x, n, \delta, r)$ is contained in the sets of points $y$ whose $(\xi, n)$-names are $\epsilon + r$-close to the $(\xi, n)$-name of $x$, i.e.

$$B(x, n, \delta, r) \subset B_{d_n^\xi}(x, \epsilon + r).$$

By Stirling's formula, it can be shown that the total number $L_n$ of such $(\xi, n)$-names consisting of $B_{d_n^\xi}(x, \epsilon + r)$ admits the following estimate:

$$L_n \le \sum_{j=0}^{[n(\epsilon+r)]} C_n^j (\#\xi - 1)^j \le \exp\big((\epsilon + r)nK_1\big),$$

where $[n(\epsilon + r)]$ denotes the largest integer no larger than $n(\epsilon + r)$, $K_1 > 1$ is a constant independent of $x$ and $n$. We want to estimate the measure of those points in $A_l$ whose $(\xi, n)$-names have an element of the partition $\xi_n := \xi \vee T^{-1}\xi \vee \cdots \vee T^{-n}\xi$ of measure greater than $\exp((-h_\mu(T, \xi) + 2K_1(\epsilon + r))n)$ in their $d_n^\xi$ $\epsilon$-neighborhood. It is obvious that the total number of such elements does not exceed $\exp((h_\mu(T, \xi) - 2K_1(\epsilon + r))n)$. Hence, the total number $Q_n$ of elements in their $d_n^\xi$ $\epsilon$-neighborhood satisfies:

$$\begin{aligned} Q_n &\le \exp\big((\epsilon + r)nK_1\big)\exp\big(\big(h_\mu(T, \xi) - 2K_1(\epsilon + r)\big)n\big) \\ &= \exp\big(\big(h_\mu(T, \xi) - K_1(\epsilon + r)\big)n\big). \end{aligned}$$

By the Shannon–McMillan–Breiman theorem for $\mu$-almost every $x \in X$, there exists $N_2(x)$ such that for any $n \ge N_2(x)$,

$$\liminf_{n\to\infty} \frac{-\log \mu(C_{\xi_n}(x))}{n} \ge h_\mu(T, \xi) - \epsilon.$$

Let $B_k := \{x \in X : N_2(x) \le k\}$. Clearly, the sets $B_k$ are nested and exhaust $X$ up to a set of measure zero. Therefore, there exists $k_0 > 1$ such that $\mu(B_k) \ge 1 - \gamma$ for any $k \ge k_0$. Fix such a number $k$ and consider those of the $Q_n$ elements of $\xi_n$ whose intersection with $A_l \cap B_k$ have positive measure. To estimate their total measure $S_n$, we multiply their number by the upper bound of their measure:

$$\begin{aligned} S_n &\le \exp\big(\big(h_\mu(T, \xi) - K_1(\epsilon + r)\big)n\big)\exp\big(-n\big(h_\mu(T, \xi) - \epsilon\big)\big) \\ &= \exp\big(-(K_1 - 1)n\epsilon - K_1 nr\big). \end{aligned}$$

This implies:

$$\mu(D_n) \le \exp\big(-(K_1 - 1)n\epsilon - K_1 nr\big),$$

where

$$D_n := \big\{x \in A_l \cap B_k : \mu\big(B(x, n, \delta, r)\big) > \exp\big(\big(-h_\mu(T, \xi) + 2K_1(\epsilon + r)\big)n\big)\exp\big((\epsilon + r)nK_1\big)\big\}.$$

Since $K_1 > 1$, we have $\sum_{n=1}^\infty \mu(D_n) < \infty$. By the Borel–Cantelli lemma, we get:

$$\mu\Big(\limsup_{n\to\infty} D_n\Big) = 0 \quad \Rightarrow \quad \mu\Big(\liminf_{n\to\infty}(X \setminus D_n)\Big) = 1.$$

This implies that for $\mu$ a.e. $x \in A_l \cap B_k$,

$$\liminf_{n\to\infty} \frac{-\log \mu(B(x, n, \delta, r))}{n} \ge h_\mu(T, \xi) - 3K_1(\epsilon + r) \ge h_\mu(T) - \epsilon - 3K_1(\epsilon + r).$$

Furthermore, we obtain:

$$\lim_{r\to 0}\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{-\log\mu(B(x,n,\delta,r))}{n}\geq h_\mu(T).$$

Since $\mu(A_l\cap B_k)\geq 1-2\gamma$ and $\gamma$ is arbitrary, we get for $\mu$ a.e. $x$,

$$\lim_{r\to 0}\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{-\log\mu(B(x,n,\delta,r))}{n}\geq h_\mu(T).$$

This completes the proof of the theorem. □

We remark that the above proof is the case of finite entropy. A slightly more complicated argument allows us to include transformations with infinite entropy as well. At last, we consider the symbolic space $(\Sigma_m^+,\sigma)$ as an example showing the measure theoretic $r$-entropy as $r$ evolves. The metric $d_\theta$ $(0<\theta<1)$ on $(\Sigma_m^+,\sigma)$ is given by

$$d_\theta(x,y)=\theta^{n(x,y)},$$

where $x=(x_0,x_1,\cdots)$, $y=(y_0,y_1,\cdots)\in\Sigma_m^+$ and $n(x,y)=\inf\{i:x_i\neq y_i\}$. Choose $\mu=\sum_{i=0}^{m-1}\frac{1}{m}\delta_i$. For any $x\in\Sigma_m^+$, and $\epsilon>0$ there exists $k:=k(\epsilon)$ such that

$$[x_0,x_1,\cdots,x_{n+k+1}]\subset B_n(x,\epsilon)\subset[x_0,x_1,\cdots,x_{n+k}].$$

Then we have:

$$\lim_{n\to\infty}\frac{-\log\mu(B(x,n,\epsilon,r))}{n}=\lim_{n\to\infty}\frac{-\log(\frac{1}{m})^{n+k}}{n}+\lim_{n\to\infty}\frac{-\log\sum_{j=0}^{[nr]}C_{n+k}^j(m-1)^j}{n}.$$

It follows from Stirling's formula that

$$\lim_{n\to\infty}\frac{-\log\sum_{j=0}^{[nr]}C_{n+k}^j(m-1)^j}{n}=-r\log(m-1)+r\log r+(1-r)\log(1-r).$$

Hence,

$$\lim_{n\to\infty}\frac{-\log\mu(B(x,n,\epsilon,r))}{n}=\log m-r\log(m-1)+r\log r+(1-r)\log(1-r).$$

If we let

$$B(g,n,x,\delta):=\left\{y\in X:\#\left\{i:d\left(T^ix,T^iy\right)<\delta\right\}=n-g(n)\right\},$$

where $g:\mathbb{N}\to\mathbb{N}$, $g(n)\leq g(n+1)\leq n+1$, $g(n)\to\infty,n\to\infty$. Then

$$\frac{-\log\mu(B(g,n,x,\epsilon))}{n}=-\frac{\log(\frac{1}{m})^{n+k}+\log C_{n+k}^{g(n)}+\log(m-1)^{g(n)}}{n}.$$

It follows from Stirling's formula that

$$C_{n+k}^{g(n)}\backsim\frac{(n+k)^{\frac{3(n+k)}{2}}}{2\pi(n+k-g(n))^{\frac{3(n+k-g(n))}{2}}g(n)^{\frac{3g(n)}{2}}}.$$

This implies that

$$\begin{aligned}\frac{-\log\mu(B(g,n,x,\epsilon))}{n}&=\frac{n+k}{n}\log m+\frac{\log 2\pi}{n}+\frac{3(n+k)}{2n}\log\frac{n+k-g(n)}{n+k}+\frac{3g(n)}{2n}\log\frac{g(n)}{n+k-g(n)}\\&\quad+\frac{g(n)}{n}\log(m-1).\end{aligned}$$

Assume that $\lim_{n\to\infty}\frac{g(n)}{n}=\alpha$, we have:

$$\lim_{n\to\infty}\frac{-\log\mu(B(g,n,x,\epsilon))}{n}=\log m+\frac{3}{2}\log(1-\alpha)+\frac{3}{2}\alpha\log\frac{\alpha}{1-\alpha}+\alpha\log(m-1).$$

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