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# Universal sampling, quasicrystals and bounded remainder sets <sup>☆</sup>



*Échantillonnage universel, quasicristaux et ensembles à restes bornés*

Sigrid Grepstad <sup>a</sup>, Nir Lev <sup>b</sup>

<sup>a</sup> Department of Mathematical Sciences, Norwegian University of Science and Technology (NTNU), NO-7491 Trondheim, Norway

<sup>b</sup> Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel

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## ABSTRACT

We examine the result due to Matei and Meyer that simple quasicrystals are universal sampling sets, in the critical case when the density of the sampling set is equal to the measure of the spectrum. We show that in this case, an arithmetical condition on the quasicrystal determines whether it is a universal set of “stable and non-redundant” sampling.

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## R É S U M É

Nous examinons le résultat, dû à Matei et à Meyer, selon lequel les quasicristaux simples sont des ensembles d'échantillonnage universel, dans le cas critique où la densité de l'ensemble d'échantillonnage est égale à la mesure du spectre. Nous montrons que, dans ce cas, une condition arithmétique sur le quasicristal détermine s'il s'agit d'un ensemble universel d'échantillonnage « stable et non redondant ».

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## 1. Introduction

1.1. A (multi-dimensional) “signal” is a function  $f \in L^2(\mathbb{R}^d)$  whose Fourier transform

$$\widehat{f}(x) = \int f(t)e^{-2\pi i(x,t)} dx$$

is supported by a bounded, measurable set  $S \subset \mathbb{R}^d$ , called “the spectrum” of the signal. We denote by  $PW_S$  the space of all  $f \in L^2(\mathbb{R}^d)$  whose Fourier transform is supported by  $S$ .

A set  $\Lambda \subset \mathbb{R}^d$  is called *uniformly discrete* if there is  $\delta(\Lambda) > 0$  such that  $|\lambda - \lambda'| > \delta(\Lambda)$  for any two distinct points  $\lambda, \lambda' \in \Lambda$ . A uniformly discrete set  $\Lambda$  is called a *set of stable sampling* for the space  $PW_S$  if the inequalities

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E-mail addresses: sigrid.grepstad@math.ntnu.no (S. Grepstad), levnir@math.biu.ac.il (N. Lev).

$$A\|f\|_{L^2(\mathbb{R}^d)} \leq \left( \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \right)^{1/2} \leq B\|f\|_{L^2(\mathbb{R}^d)}$$

hold for every  $f \in PW_S$  with positive constants  $A, B$  not depending on  $f$ . We call  $\Lambda$  a *set of stable interpolation* for  $PW_S$  if for every sequence  $\{c_\lambda\} \in \ell^2(\Lambda)$  there exists at least one  $f \in PW_S$  such that  $f(\lambda) = c_\lambda$  for all  $\lambda \in \Lambda$ .

A general problem is to determine when  $\Lambda$  is a sampling or interpolation set for  $PW_S$ . An important role in this problem is played by the lower and upper uniform densities

$$D^-(\Lambda) = \liminf_{R \rightarrow \infty} \inf_{x \in \mathbb{R}^d} \frac{\#(\Lambda \cap (x + B_R))}{|B_R|},$$

$$D^+(\Lambda) = \limsup_{R \rightarrow \infty} \sup_{x \in \mathbb{R}^d} \frac{\#(\Lambda \cap (x + B_R))}{|B_R|},$$

where  $B_R$  denotes the ball of radius  $R$  centered at the origin. Landau [9] (see also [13]) obtained necessary conditions for sampling and interpolation in terms of these densities:

If  $\Lambda$  is a sampling set for  $PW_S$ , then  $D^-(\Lambda) \geq \text{mes } S$ ;

If  $\Lambda$  is an interpolation set for  $PW_S$ , then  $D^+(\Lambda) \leq \text{mes } S$ .

In the case when  $S$  is an interval on  $\mathbb{R}$ , this is due to Beurling [2] and Kahane [7], who also proved that the condition  $D^-(\Lambda) > \text{mes } S$  is sufficient for sampling, while the condition  $D^+(\Lambda) < \text{mes } S$  is sufficient for interpolation. However, for disconnected spectra and in the multi-dimensional case, sufficient conditions in terms of these densities can no longer be given.

For a “regularly distributed” set  $\Lambda$ , the two densities  $D^-(\Lambda)$  and  $D^+(\Lambda)$  coincide. In this case, their common value is called the *uniform density* of  $\Lambda$  and will be denoted by  $D(\Lambda)$ .

1.2. Olevskii and Ulanovskii established the existence of “universal” sets  $\Lambda$ , for which the sampling and interpolation problem does admit sufficient conditions in terms of density.

**Theorem.** ([14–16]) *Given  $a > 0$  there is a uniformly discrete set  $\Lambda$ ,  $D(\Lambda) = a$ , such that*

- (i)  $\Lambda$  is a sampling set for  $PW_S$  for any compact set  $S \subset \mathbb{R}^d$  with  $\text{mes } S < D(\Lambda)$ ;
- (ii)  $\Lambda$  is an interpolation set for  $PW_S$  for any open set  $S \subset \mathbb{R}^d$  with  $\text{mes } S > D(\Lambda)$ .

A set  $\Lambda$  with the property (i) above is called a “universal sampling set”, while a set satisfying property (ii) is a “universal interpolation set”. It was shown in [14–16] that such a set  $\Lambda$  may be constructed by an arbitrarily small perturbation of a lattice in  $\mathbb{R}^d$ .

1.3. A different construction of universal sampling and interpolation sets, based on Meyer’s “cut-and-project” method, was presented by Matei and Meyer in [11,12].

Let  $\Gamma$  be a lattice in  $\mathbb{R}^{d+1} = \mathbb{R}^d \times \mathbb{R}$ , and let  $p_1$  and  $p_2$  denote the projections onto  $\mathbb{R}^d$  and  $\mathbb{R}$ , respectively. We assume that the restrictions of  $p_1$  and  $p_2$  to  $\Gamma$  are injective, and that their images are dense. Let  $I = [a, b)$  be a semi-closed interval on  $\mathbb{R}$ , called a “window”, and consider the cut-and-project set  $\Lambda$  in  $\mathbb{R}^d$  defined by

$$\Lambda = \Lambda(\Gamma, I) = \{p_1(\gamma) : \gamma \in \Gamma, p_2(\gamma) \in I\}. \quad (1)$$

Such a set is called a “simple quasicrystal”. One can show that  $\Lambda$  is uniformly discrete and

$$D(\Lambda) = \frac{|I|}{\det \Gamma}.$$

**Theorem.** ([11,12]) *If  $\Lambda$  is a simple quasicrystal defined by (1) then it satisfies (i) and (ii).*

The result does not cover the critical case when the density of  $\Lambda$  is equal to the measure of the spectrum  $S$ . The aim of the present note is to examine this critical case. In one dimension and in the periodic setting, such results were obtained by Kozma and Lev in [8].

1.4. A bounded set  $S \subset \mathbb{R}^d$  is called *Riemann measurable* if its boundary has measure zero. We say that two Riemann measurable sets  $S$  and  $S'$  in  $\mathbb{R}^d$  are *equidecomposable*, or *scissors congruent*, if the set  $S$  can be partitioned into finitely many Riemann measurable subsets that can be reassembled by rigid motions to form, up to measure zero, a partition of  $S'$ .

The notion of equidecomposability goes back to Hilbert’s third problem – the question of whether two polyhedra of equal volume are necessarily equidecomposable (see [3] for a detailed exposition of the subject).

Sometimes it is interesting to consider a restricted notion of equidecomposability, where the pieces of the partition are allowed to be reassembled only by motions belonging to some given subgroup of all the rigid motions. We will consider equidecomposability with respect to the group of translations by vectors belonging to  $p_1(\Gamma^*)$  (a countable, dense subgroup of all the translations of  $\mathbb{R}^d$ ), where by  $\Gamma^*$  we denote the lattice dual to  $\Gamma$ .

**Theorem 1.** *Let  $\Lambda$  be a simple quasicrystal defined by (1) and satisfying the condition*

$$|I| \in p_2(\Gamma). \tag{2}$$

*Then  $\Lambda$  is simultaneously a sampling and interpolation set for  $PW_S$  for every Riemann measurable set  $S$ ,  $\text{mes } S = D(\Lambda)$ , which satisfies the following condition: (\*)  $S$  is equidecomposable to a parallelepiped with vertices in  $p_1(\Gamma^*)$ , using translations by vectors belonging to  $p_1(\Gamma^*)$ .*

The collection of admissible sets  $S$  in this result is, in a sense, “dense” among the sets of measure  $D(\Lambda)$ . Indeed, let a compact set  $K$  and an open set  $U$  in  $\mathbb{R}^d$  be given, satisfying  $K \subset U$  and  $\text{mes } K < D(\Lambda) < \text{mes } U$ . Then, assuming (2), one can find a set  $S$  satisfying (\*) such that  $K \subset S \subset U$  and  $\text{mes } S = D(\Lambda)$ . We may therefore say that  $\Lambda$  is a “universal set of simultaneous sampling and interpolation”.

Interest in simultaneous sampling and interpolation sets is partly due to the fact that these are precisely the “non-redundant” sampling sets, namely sampling sets that are minimal with respect to inclusion (see [17]). Universal sets of this type were first constructed in [14–16].

We mention that special cases of Theorem 1 were previously obtained in [10] and [4].

The role of the arithmetical condition (2) imposed in Theorem 1 is clarified by the following

**Theorem 2.** *Let  $\Lambda$  be a simple quasicrystal defined by (1) and such that*

$$|I| \notin p_2(\Gamma). \tag{3}$$

*Then there does not exist any Riemann measurable set  $S$  such that  $\Lambda$  is simultaneously a sampling and interpolation set for  $PW_S$ .*

1.5. By particular choices of the lattice  $\Gamma$  and the window  $I$  one can obtain the following examples as special cases of Theorem 1.

**Example 1.** Let  $\alpha$  be an irrational number, and define

$$\lambda(n) = n + \{n\alpha\}, \quad n \in \mathbb{Z}$$

(where  $\{x\}$  denotes the fractional part of  $x$ ). Then the sequence  $\Lambda = \{\lambda(n)\}$  is simultaneously a sampling and interpolation set for  $PW_S$  for every set  $S \subset \mathbb{R}$  which is a finite union of disjoint intervals with lengths in  $\mathbb{Z}\alpha + \mathbb{Z}$  and of total length 1.

**Example 2.** The sequence  $\Lambda = \{\lambda(n, m)\}$  defined by

$$\lambda(n, m) = (n, m) + \{n\sqrt{2} + m\sqrt{3}\}(\sqrt{2}, \sqrt{3}), \quad (n, m) \in \mathbb{Z}^2,$$

is simultaneously a sampling and interpolation set for  $PW_S$  for every set  $S \subset \mathbb{R}^2$  which is equidecomposable to the unit cube  $Q = [0, 1]^2$  using translations by vectors in  $\mathbb{Z}(\sqrt{2}, \sqrt{3}) + \mathbb{Z}^2$ .

## 2. Bounded remainder sets

2.1. By applying a linear transformation on  $\mathbb{R}^d \times \mathbb{R}$  it would be enough to consider the case when

$$\Gamma = \{((\text{Id} + \beta\alpha^\top)m - \beta n, n - \alpha^\top m) : m \in \mathbb{Z}^d, n \in \mathbb{Z}\}, \tag{4}$$

$$\Gamma^* = \{(m + \alpha n, (1 + \beta^\top\alpha)n + \beta^\top m) : m \in \mathbb{Z}^d, n \in \mathbb{Z}\}, \tag{5}$$

where  $\text{Id}$  denotes the  $d \times d$  identity matrix,  $\alpha$  and  $\beta$  are column vectors in  $\mathbb{R}^d$ , the vector  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)^\top$  is such that the numbers  $1, \alpha_1, \alpha_2, \dots, \alpha_d$  are linearly independent over the rationals, and the vector  $\beta = (\beta_1, \beta_2, \dots, \beta_d)^\top$  is such that the numbers  $\beta_1, \beta_2, \dots, \beta_d, 1 + \beta^\top\alpha$  are linearly independent over the rationals.

Note that  $\det \Gamma = 1$  and hence the quasicrystal  $\Lambda = \Lambda(\Gamma, I)$  has uniform density  $D(\Lambda) = |I|$ .

2.2. For a bounded, measurable set  $S$  in  $\mathbb{R}^d$ , let

$$\chi_S(x) = \sum_{k \in \mathbb{Z}^d} \mathbb{1}_S(x+k)$$

denote the multiplicity function of the projection of  $S$  on  $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ . We say that  $S$  is a *bounded remainder set* (BRS) with respect to the vector  $\alpha$  if there is a constant  $C = C(S, \alpha)$  such that

$$\left| \sum_{k=0}^{n-1} \chi_S(x+k\alpha) - n \operatorname{mes} S \right| \leq C \quad (n = 1, 2, 3, \dots) \quad \text{for a.e. } x \in \mathbb{T}^d.$$

The study of bounded remainder sets is a classical topic in the theory of discrepancy, see [5] and the references therein.

**Theorem 3.** *The simple quasicrystal  $\Lambda(\Gamma, I)$  is simultaneously a sampling and interpolation set for  $PW_S$  for every Riemann measurable bounded remainder set  $S$ ,  $\operatorname{mes} S = |I|$ .*

2.3. The condition (2) in Theorem 1 says that there are integers  $n_0, n_1, \dots, n_d$  such that

$$|I| = n_0 + n_1\alpha_1 + \dots + n_d\alpha_d. \quad (6)$$

We proved in [5] that in this case, there is a parallelepiped  $P$  with vertices in  $\mathbb{Z}\alpha + \mathbb{Z}^d$  such that  $\operatorname{mes} P = |I|$ . We also proved there that any such a parallelepiped is a bounded remainder set, as well as any set  $S$  which is equidecomposable to  $P$  using translations by vectors in  $\mathbb{Z}\alpha + \mathbb{Z}^d$ . Hence Theorem 1 follows from Theorem 3 and the results in [5].

We remark that the equidecomposability to a parallelepiped with vertices in  $\mathbb{Z}\alpha + \mathbb{Z}^d$ , using translations by vectors in  $\mathbb{Z}\alpha + \mathbb{Z}^d$ , is in fact a characterization of the Riemann measurable bounded remainder sets [5].

### 3. Duality

A key principle in [11,12] is a “duality” connecting sampling and interpolation from the simple quasicrystal  $\Lambda(\Gamma, I)$  to another (non-simple) quasicrystal in  $\mathbb{R}$  defined by

$$\Lambda^*(\Gamma, S) = \{p_2(\gamma^*) : \gamma^* \in \Gamma^*, p_1(\gamma^*) \in S\}.$$

We may assume that the boundary of the Riemann measurable set  $S$  does not contain any point belonging to  $p_1(\Gamma^*)$ . The duality principle can then be formulated as follows.

**Lemma 1.**  *$\Lambda(\Gamma, I)$  is a sampling set (respectively, an interpolation set) for  $PW_S$  if and only if  $\Lambda^*(\Gamma, S)$  is an interpolation set (respectively, a sampling set) for  $PW_I$ .*

This allows us to reduce the problem to the single interval  $I$ . The proof of Lemma 1 is along similar lines as in [12, Sections 6–7]. See also [8, Section 2].

### 4. Proof of Theorem 3

We assume (4), (5) and that  $S$  is a Riemann measurable bounded remainder set with respect to  $\alpha$ ,  $\operatorname{mes} S = |I|$ . To prove that the quasicrystal  $\Lambda(\Gamma, I)$  is a sampling and interpolation set for  $PW_S$  it would be enough, by Lemma 1, to show that the dual quasicrystal  $\Lambda^*(\Gamma, S)$  is a sampling and interpolation set for  $PW_I$ .

We now follow the approach used in [4,8,10]. By a theorem of Avdonin [1], for  $\Lambda^*(\Gamma, S)$  to be a sampling and interpolation set for  $PW_I$  it is sufficient that for some enumeration  $\{\lambda_j, j \in \mathbb{Z}\}$  of the set  $\Lambda^*(\Gamma, S)$ , the following three conditions are satisfied:

- (a)  $\{\lambda_j\}$  is a separated sequence, that is,  $\inf_{j \neq k} |\lambda_j - \lambda_k| > 0$ ;
- (b)  $\sup_j |\delta_j| < \infty$ , where  $\delta_j = \lambda_j - j/|I|$ ;
- (c) there is a constant  $c$  and a positive integer  $N$  such that

$$\sup_{k \in \mathbb{Z}} \left| \frac{1}{N} \sum_{j=k+1}^{k+N} \delta_j - c \right| < \frac{1}{4|I|}. \quad (7)$$

Define

$$S_n = S \cap (n\alpha + \mathbb{Z}^d), \quad A_n = \{n + \langle x, \beta \rangle : x \in S_n\}, \quad n \in \mathbb{Z}.$$

Then the system  $\{\Lambda_n\}$  forms a partition of  $\Lambda^*(\Gamma, S)$  (where some of the sets  $\Lambda_n$  may be empty). Let  $\{s_n\}$  be a sequence of integers such that

$$s_{n+1} - s_n = \#\Lambda_n, \tag{8}$$

and choose an enumeration  $\{\lambda_j, j \in \mathbb{Z}\}$  of the set  $\Lambda^*(\Gamma, S)$  such that

$$\Lambda_n = \{\lambda_j : s_n \leq j < s_{n+1}\}, \quad n \in \mathbb{Z}.$$

We will show that for such an enumeration, conditions (a), (b) and (c) above are satisfied.

Condition (a) obviously holds since  $\Lambda^*(\Gamma, S)$  is a uniformly discrete set.

To confirm conditions (b) and (c) we need the following lemma.

**Lemma 2.** (See [5]) *If  $S$  is a Riemann measurable bounded remainder set, then there is a bounded, Riemann integrable “transfer function”  $g : \mathbb{T}^d \rightarrow \mathbb{R}$  satisfying*

$$\chi_S(x) - \text{mes } S = g(x + \alpha) - g(x) \tag{9}$$

for a.e.  $x \in \mathbb{T}^d$ .

By an appropriate translation of  $S$  we may assume that (9) holds for all  $x \in \mathbb{Z}\alpha$ . Since we have  $s_{n+1} - s_n = \#\Lambda_n = \chi_S(n\alpha)$ , this implies that

$$s_n = n \text{mes } S + g(n\alpha) + \text{const} = n|I| + O(1),$$

from which (b) can be deduced easily.

To establish (c) one can show as in [10] that

$$\sum_{s_n \leq j < s_{n+1}} \delta_j = h(n\alpha), \quad n \in \mathbb{Z},$$

for an appropriate Riemann integrable function  $h : \mathbb{T}^d \rightarrow \mathbb{R}$ . Since the points  $\{n\alpha\}$  are well-distributed on  $\mathbb{T}^d$ , we have

$$\sup_{k \in \mathbb{Z}} \left| \frac{1}{N} \sum_{n=k+1}^{k+N} h(n\alpha) - \int_{\mathbb{T}^d} h(x) dx \right| = o(1), \quad N \rightarrow \infty.$$

This implies that (7) holds provided that  $N$  is sufficiently large, with

$$c = \frac{1}{\text{mes } S} \int_{\mathbb{T}^d} h(x) dx.$$

### 5. Proof of Theorem 2

Again we assume (4), (5), and suppose that  $\Lambda(\Gamma, I)$  is simultaneously a sampling and interpolation set for  $PW_S$ , where  $S$  is a Riemann measurable set. We will show that in this case (3) cannot hold.

Here the approach is similar to the one in [8]. By Landau’s inequalities we have  $\text{mes } S = |I|$ . By Lemma 1, the dual quasicrystal  $\Lambda^*(\Gamma, S)$  is a sampling and interpolation set for  $PW_I$ . According to a theorem of Pavlov (see [6, p. 240]), a necessary condition for this is that

$$f(x) = n(x) - |I|x$$

is a function in  $\text{BMO}(\mathbb{R})$ , where  $n(x)$  is the “counting function” of the set  $\Lambda^*(\Gamma, S)$  satisfying

$$n(y) - n(x) = \#(\Lambda^*(\Gamma, S) \cap [x, y])$$

(which is defined uniquely up to an additive constant). In turn, this implies that the sequence

$$\{s_n - n \text{mes } S\}$$

belongs to  $\text{BMO}(\mathbb{Z})$ , where  $\{s_n\}$  is again the sequence defined by (8). But since

$$s_n = \sum_{k=0}^{n-1} \chi_S(k\alpha) + \text{const} \quad (n = 1, 2, 3, \dots)$$

it follows from the results in [8, Section 4] that there are integers  $n_0, n_1, \dots, n_d$  such that

$$\text{mes } S = n_0 + n_1\alpha_1 + \dots + n_d\alpha_d.$$

Thus we obtain (6), and so (2) must hold.

## 6. Remarks

1. As is well known, the simultaneous sampling and interpolation property of a set  $\Lambda$  for  $PW_S$  is equivalent to the condition that the exponential system

$$E(\Lambda) = \{e^{2\pi i(\lambda, x)}\}_{\lambda \in \Lambda}$$

is a *Riesz basis* in the space  $L^2(S)$  (see [17]). Hence our results may also be formulated in terms of the Riesz basis property of the system of exponentials with frequencies belonging to the simple quasicrystal  $\Lambda(\Gamma, I)$ .

2. The results admit analogous versions in the periodic (multi-dimensional) setting, which can be proved in a similar way. The details will be published elsewhere.

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