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Complex analysis/Harmonic analysis

A characterization of Möbius transformations

Une caractérisation des transformations de Möbius

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ABSTRACT

We prove that the derivative θ' of an inner function θ is outer if and only if θ is a Möbius transformation. An alternative characterization involving a reverse Schwarz–Pick type estimate is also given.

RÉSUMÉ

Étant donnée une fonction intérieure θ, on démontre que sa dérivée θ' est extérieure si et seulement si θ est une transformation de Möbius.

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1. Introduction and main result

Let H^∞ stand for the algebra of bounded holomorphic functions on the disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. A function $\theta \in H^\infty$ is called *inner* if $\lim_{r\to 1^-} |\theta(r\zeta)| = 1$ at almost every point ζ of the circle $\mathbb{T} := \partial\mathbb{D}$. Among the nonconstant inner functions, the simplest ones are undoubtedly the conformal automorphisms of the disk, also known as *Möbius transformations*; these are of the form

$$\theta_{\lambda,a}(z) := \lambda \frac{z-a}{1-\bar{a}z}$$

for some $\lambda \in \mathbb{T}$ and $a \in \mathbb{D}$. A calculation shows that

$$\theta'_{\lambda,a}(z) = \lambda \frac{1-|a|^2}{(1-\bar{a}z)^2},$$

which happens to be an *outer* function. (A nonvanishing holomorphic function f on $\mathbb{D}$ is said to be outer if $\log|f|$ agrees with the harmonic extension of an integrable function on $\mathbb{T}$.)

In this note, we prove that the property of θ' being outer actually characterizes the Möbius transformations among all inner functions θ.

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Before stating the result rigorously, we need to recall that the *Nevanlinna class* $\mathcal{N}$ (resp., the *Smirnov class* $\mathcal{N}^+$) is formed by the functions that can be written as u/v, where $u, v \in H^\infty$ and v is zero-free (resp., outer) on $\mathbb{D}$. The reader is referred to [5, Chapter II] for further information on $\mathcal{N}$ and $\mathcal{N}^+$, including the canonical factorization theorem for functions from these spaces. We also mention the fact that, for θ inner, one has $\theta' \in \mathcal{N}$ if and only if $\theta' \in \mathcal{N}^+$; see [1] for a proof. In what follows, we are forced to require that θ' be in $\mathcal{N}$ (or $\mathcal{N}^+$), since this is apparently the weakest natural assumption that allows us to speak of the inner–outer factorization for θ'.

Theorem 1.1. *Let θ be a nonconstant inner function with $\theta' \in \mathcal{N}$. Then θ' is outer if and only if θ is a Möbius transformation.*

In some special cases, the fact that the derivative of a non-Möbius inner function will have a nontrivial inner part may be obvious or due to known results. First of all, θ' will certainly vanish at the multiple zeros of θ, if any. Secondly, a result of Ahern and Clark (see [1, Corollary 4]) tells us that the singular factor of θ, if existent, gets inherited by θ', provided that the latter function is in $\mathcal{N}$. Thus, in a sense, singular factors can be thought of as responsible for the (boundary) zeros of infinite multiplicity. Thirdly, if θ is a finite Blaschke product with at least two zeros, then θ' is sure to have zeros in $\mathbb{D}$ (see [6] for a more precise information on the location of these), so θ' will again be non-outer. The remaining case, where θ is an infinite Blaschke product with simple zeros, seems however to be new.

2. Proof of Theorem 1.1

To prove the nontrivial part of the theorem, assume that θ is inner and θ' is an outer function in $\mathcal{N}$.

For all $z \in \mathbb{D}$ and almost all $\zeta \in \mathbb{T}$, Julia's lemma (see [2] or [5, p. 41]) yields

$$\frac{|\theta(\zeta)-\theta(z)|^2}{1-|\theta(z)|^2} \le \left|\theta'(\zeta)\right| \cdot \frac{|\zeta - z|^2}{1-|z|^2}, \tag{2.1}$$

or equivalently,

$$\frac{1-|z|^2}{1-|\theta(z)|^2} \cdot \left|\frac{1-\overline{\theta(z)}\theta(\zeta)}{1-\bar{z}\zeta}\right|^2 \le \left|\theta'(\zeta)\right|. \tag{2.2}$$

Further, we associate with every (fixed) $z \in \mathbb{D}$ the H^∞-function

$$\Phi_z(w) := \frac{1-|z|^2}{1-|\theta(z)|^2} \cdot \left(\frac{1-\overline{\theta(z)}\theta(w)}{1-\bar{z}w}\right)^2 \tag{2.3}$$

and rewrite (2.2) in the form

$$\left|\Phi_z(\zeta)\right| \le \left|\theta'(\zeta)\right|, \quad \zeta \in \mathbb{T}. \tag{2.4}$$

Since $\Phi_z \in H^\infty$ and θ' is outer, the ratio $\psi_z := \Phi_z/\theta'$ will be in $\mathcal{N}^+$; and since, by (2.4), $|\psi_z| \le 1$ a.e. on $\mathbb{T}$, it follows that ψ_z is in H^∞ and has norm at most 1. In other words, the estimate (2.4) extends into the disk, so that

$$\left|\Phi_z(w)\right| \le \left|\theta'(w)\right|, \quad w \in \mathbb{D}.$$

In particular, putting $w = z$, we obtain

$$\left|\Phi_z(z)\right| \le \left|\theta'(z)\right|. \tag{2.5}$$

A glance at (2.3) reveals that

$$\left|\Phi_z(z)\right| = \Phi_z(z) = \frac{1-|\theta(z)|^2}{1-|z|^2},$$

and plugging this into (2.5) gives

$$\frac{1-|\theta(z)|^2}{1-|z|^2} \le \left|\theta'(z)\right|.$$

In conjunction with the Schwarz–Pick estimate

$$\left|\theta'(z)\right| \le \frac{1-|\theta(z)|^2}{1-|z|^2} \tag{2.6}$$

(see [5, Chapter I, Section 1]), this means that we actually have equality in (2.6). This last fact is known to imply that θ is a Möbius transformation (see *ibid*), and the proof is complete.

3. An alternative characterization and open questions

The primary purpose of this note, essentially accomplished by now, can be described as giving a short and self-contained proof of a result from [4]. In that paper, our main concern was a certain reverse Schwarz–Pick type inequality for unit-norm H^∞ functions (see also [3] for an earlier version); the above characterization of Möbius transformations was then deduced as a corollary. In addition, it was shown in [4, Section 2] that, among the nonconstant inner functions θ with $\theta' \in \mathcal{N}$, the Möbius transformations are also characterized by the property that

$$\eta\left(\frac{1-|\theta(z)|^2}{1-|z|^2}\right) \le \left|\theta'(z)\right|, \quad z \in \mathbb{D}, \tag{3.1}$$

for some nondecreasing function $\eta : (0,\infty) \to (0,\infty)$. We now improve this last result by relaxing the *a priori* assumptions on θ. In fact, it turns out that we need not restrict our attention to inner functions from the start. Instead, we shall verify that θ will have to be inner (and with derivative in $\mathcal{N}$) automatically, under the milder hypotheses below.

Proposition 3.1. *Let $\theta \in H^\infty$ be a nonconstant function with $\|\theta\|_\infty \le 1$. The following conditions are equivalent:*

(i) *θ is a Möbius transformation,*
(ii) *there is a nondecreasing function $\eta : (0,\infty) \to (0,\infty)$ with $\lim_{t\to\infty} \eta(t) = \infty$ making (3.1) true.*

Proof. Of course, (i) implies (ii) with $\eta(t) = t$. To prove the nontrivial implication (ii) ⇒ (i), observe that

$$\inf\left\{\frac{1-|\theta(z)|^2}{1-|z|^2} : z \in \mathbb{D}\right\} > 0$$

(by Schwarz's lemma), and so (3.1) yields $\inf_{z\in\mathbb{D}} |\theta'(z)| > 0$. Hence $1/\theta' \in H^\infty$ and $\theta' \in \mathcal{N}$; in particular, θ' has radial limits almost everywhere on $\mathbb{T}$.

Now, if $\zeta \in \mathbb{T}$ is a point at which $\lim_{r\to 1^-} |\theta(r\zeta)| < 1$, then (3.1) shows that $\lim_{r\to 1^-} |\theta'(r\zeta)| = \infty$. Consequently, the set of such ζ's has zero measure on $\mathbb{T}$. It follows that θ has radial limits of modulus 1 almost everywhere, and is therefore an inner function. To complete the proof, it remains to invoke the above-mentioned result from [4]. □

We conclude by mentioning two open questions that puzzle us. First, we would like to know which inner functions I can be written as $I = \mathrm{inn}(\theta')$ (where "inn" stands for "the inner factor of"), as θ ranges over the nonconstant inner functions with $\theta' \in \mathcal{N}$. Does every inner I arise in this way?

To pose the other question, let us introduce the notation $\sigma(I)$ for the *boundary spectrum* of an inner function I. Thus, $\sigma(I)$ is the smallest closed set $E \subset \mathbb{T}$ such that I is analytic across $\mathbb{T} \setminus E$. Now, if θ is inner (and nonconstant) with $\theta' \in \mathcal{N}$, and if $I = \mathrm{inn}(\theta')$, then it is easy to see that $\sigma(I) \subset \sigma(\theta)$. Do we actually have $\sigma(I) = \sigma(\theta)$? An affirmative answer seems plausible to us, but so far, we have only succeeded in verifying it under an additional hypothesis.

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