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Topological gradient for fourth-order PDE and application to the detection of fine structures in 2D images

Gradient topologique pour des EDP du quatrième ordre et application à la détection de structures fines dans des images 2D

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ABSTRACT

In this paper we describe a new approach for the detection of fine structures in an image. This approach is based on the computation of the topological gradient associated with a cost function defined from a regularization of the data (possibly noisy). We get this approximation by solving a fourth-order PDE. The study of the topological sensitivity is made in the cases of both a circular inclusion and a crack. We illustrate our approach by giving two experimental results.

RÉSUMÉ

Dans cette note, on décrit une nouvelle approche pour la détection de structures fines dans une image. Cette approche est basée sur le calcul du gradient topologique associé à une fonction coût définie à partir des dérivées secondes d'une régularisation des données (éventuellement bruitées). Cette régularisation est obtenue via la résolution d'une EDP du quatrième ordre. L'étude de la sensibilité topologique est faite dans les cas d'une inclusion circulaire et d'un crack. Nous illustrons notre approche en donnant deux résultats expérimentaux.

Version française abrégée

Dans cette note, on propose un nouveau modèle pour la détection de structures fines (points et filaments) dans une image définie sur un domaine $\Omega \subset \mathbb{R}^2$. Notre modèle est basé sur l'étude de la sensibilité topologique d'une fonction coût $j(\Omega) = J(\Omega, u_\Omega)$, où u_Ω est la solution d'une EDP du quatrième ordre. Nous étudions cette sensibilité lorsqu'on perturbe Ω en lui enlevant des objets ω_ϵ de taille ϵ. Les exemples les plus simples adaptés à l'étude de structures fines sont $\omega_\epsilon = B(x_0, \epsilon)$ la boule de centre x_0 et de rayon ϵ et $\omega_\epsilon = x_0 + \epsilon\sigma(n)$, où $\sigma(n)$ est un segment de droite de normale n. Nous montrons pour ces exemples que $j(\Omega_\epsilon)$ admet le développement asymptotique (1) où $\mathcal{I}(x_0)$ est, par définition, le gradient

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topologique au point x_0. Par conséquent, si on veut minimiser $j(\Omega_\epsilon)$ il est souhaitable d'insérer des petites boules ou des cracks aux points x_0 où $\mathcal{I}(x_0)$ est « le plus négatif » possible. Du point de vue de l'analyse d'images, ces petites structures correspondent aux structures fines qu'on veut détecter. La question est : quelles fonctions coût et EDP doit-on utiliser pour détecter de telles structures ? Si pour la détection de contours d'objets dans une image il est classique d'utiliser des opérateurs construits à partir du gradient spatial, ce choix est complètement inopérant pour la détection de structures fines : le gradient « ne voit pas » ces structures. Pour détecter ce type de structure, il est connu [10,5] qu'il faut utiliser des opérateurs différentiels d'ordre 2. Nous avons choisi pour cela la fonction coût définie en (2), liée à la modélisation de l'équilibre d'une plaque mince soumise à des forces extérieures. Le calcul très technique du gradient topologique revient à estimer la différence $J_\epsilon(u_\epsilon) - J_0(u_0)$ où $J_\epsilon(u) = J_{\Omega_\epsilon}(u)$. u_ϵ est l'unique solution du problème variationnel $a_\epsilon(u_\epsilon, v) = l_\epsilon(v)$, pour tout v dans $H(\Omega_\epsilon) = \{u \in L^2(\Omega_\epsilon), \nabla^2 u \in L^2(\Omega_\epsilon)\}$ où $a_\epsilon(u, v)$ et $l_\epsilon(v)$ sont définis respectivement par (3) et (4). L'expression du gradient topologique est donnée en (16), (15) pour le cas du crack et (17) pour le cas d'une inclusion circulaire. On conclut cette note en illustrant l'approche par deux résultats expérimentaux.

1. Introduction

In medical, biological or satellite imaging, the detection of fine structures (filaments, particles...) is an important task. In this note we show how it is possible to solve this problem by using topological optimization tools. Initially used for the detection of cracks [2], the notion of topological gradient has been recently applied in imaging problems such as the restoration, the segmentation, or the classification of images [7,3]. Roughly speaking, the topological gradient approach performs as follows: let Ω be an open bounded set of $\mathbb{R}^2$ and $j(\Omega) = J(\Omega, u_\Omega)$ be a cost function where u_Ω is the solution of a given PDE. For small $\epsilon > 0$, let (a) $\Omega_\epsilon = \Omega \backslash \overline{\{x_0 + \epsilon\omega\}}$ or (b) $\Omega_\epsilon = \Omega \backslash \overline{\{x_0 + \epsilon\sigma(n)\}}$, where $x_0 \in \Omega$, $\omega = B(O, 1)$ is the unit ball of $\mathbb{R}^2$ and $\sigma(n)$ is a straight segment with normal n (a crack). The topological sensitivity provides an asymptotic expansion of $j(\Omega_\epsilon)$ when $\epsilon \to 0$. In many cases it takes the form:

$$j(\Omega_\epsilon) = j(\Omega) + \epsilon^2 \mathcal{I}(x_0) + o(\epsilon^2) \tag{1}$$

$\mathcal{I}(x_0)$ is called the topological gradient at x_0. Thus if we want to minimize $j(\Omega_\epsilon)$, it would be preferable to insert small balls or cracks at points x_0 where $\mathcal{I}(x_0)$ is "the most negative". In imaging applications, the main question is the choice of the cost function and the PDE that u_Ω has to verify.

If for edge detection the usual spatial gradient is classically used, it is totally inefficient for the detection of points or filaments. It "does not see" these structures. It is known [10,5] that for the detection of fine structures we need to use algorithms based on second-order derivatives. In Section 2 we introduce the cost function $j(\Omega_\epsilon)$ and the variational formulation for u_Ω. Then in Section 3 we give the expression of the associated topological gradient in cases (a) and (b). In Section 4, we briefly describe the numerical approximation of the problem and we conclude by giving two experimental results.

2. Definition of the cost function and of the variational formulation for the detection of fine structures

Let $\Omega \subset \mathbb{R}^2$ be the image domain and $f \in L^2(\Omega)$ the grey levels of a 2-D image for which we want to detect fine structures that we liken to points or straight lines. The simplest choice of the cost function would be $J_\Omega(u) = \int_\Omega |\nabla^2 u|^2$, but for eventual other applications, we introduce a more general cost function stemming from the Kirchhoff model in the theory of thin plates ($0 \le \nu < 1$ denotes the Poisson ratio):

$$J_\Omega(u) = \int_\Omega \left((\Delta u)^2 + 2(1-\nu)\left(\left(\frac{\partial^2 u}{\partial x_1 \partial x_2} \right)^2 - \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 u}{\partial x_2^2} \right) \right) \mathrm{d}x \tag{2}$$

We remark that for $\nu = 0$ we retrieve the square of the norm of the Hessian matrix. In the sequel, Ω_ϵ is defined as in (a) or (b); for $\epsilon = 0$, $\Omega_0 = \Omega$. We denote $u_\epsilon = u_{\Omega_\epsilon}$. Now, for the definition of u_ϵ, in order to study the variations of $J_{\Omega_\epsilon}(u_\epsilon)$ it is coherent and natural to search for $u_\epsilon = \operatorname{argmin}_{u \in H(\Omega_\epsilon)}(J_{\Omega_\epsilon}(u) + \|f - u\|^2_{L^2(\Omega_\epsilon)})$, with $H(\Omega_\epsilon) = \{u \in L^2(\Omega_\epsilon), \nabla^2 u \in L^2(\Omega_\epsilon)\}$, or equivalently as the solution of the variational problem:

$$\text{find } u_\epsilon \in H(\Omega_\epsilon) \text{ such that } a_\epsilon(u_\epsilon, v) = l_\epsilon(v), \quad \forall v \in H(\Omega_\epsilon)$$

$$a_\epsilon(u, v) = \int_{\Omega_\epsilon} \Delta u \Delta v + (1-\nu)\left(2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 v}{\partial x_1 \partial x_2} - \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 v}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 v}{\partial x_1^2} \right) + uv \tag{3}$$

$$l_\epsilon(v) = \int_{\Omega_\epsilon} f v \, \mathrm{d}x \tag{4}$$

Thanks to the Lax–Milgram lemma, u_ϵ is unique in $H(\Omega_\epsilon)$. Moreover, u_ϵ satisfies the Euler equation:

$$\begin{cases} \Delta^2 u_\epsilon + u_\epsilon = f, & \text{in } \Omega_\epsilon \\ B_1(u_\epsilon) = 0, \quad B_2(u_\epsilon) = 0 & \text{on } \partial\Omega_\epsilon \end{cases} \tag{5}$$

with

$$\begin{cases} B_1(u) = \partial_n(\Delta u) - (1-\nu)\partial_\sigma\left(n_1 n_2\left(\dfrac{\partial^2 u}{\partial x_1^2} - \dfrac{\partial^2 u}{\partial x_2^2}\right) - \left(n_1^2 - n_2^2\right)\dfrac{\partial^2 u}{\partial x_1 \partial x_2}\right) \\ B_2(u) = \nu \Delta u + (1-\nu)\left(n_1^2 \dfrac{\partial^2 u}{\partial x_1^2} + n_2^2 \dfrac{\partial^2 u}{\partial x_2^2} + 2 n_1 n_2 \dfrac{\partial^2 u}{\partial x_1 \partial x_2}\right) \end{cases} \tag{6}$$

where $n = (n_1, n_2)$ is the outside normal to $\partial\Omega_\epsilon$ and $\sigma = (\sigma_1, \sigma_2)$ is the tangent vector with $|n| = |\sigma| = 1$.

Remark 1. The study of topological sensitivity for fourth-order operators is not new. In [1], the authors, in a different context, compute the topological gradient for the Kirchhoff plate bending problem in case (a) of a circular inclusion. Our model is simpler and we are able to give explicit expressions of the topological gradients in the cases of both circular inclusions and cracks.

3. Computation of the topological gradient associated with $J_{\Omega_\epsilon}(u_\epsilon)$

We denote $J_\epsilon(u) = J_{\Omega_\epsilon}(u)$, and to shorten the presentation we only give the main idea in case (b): $\Omega_\epsilon = \Omega \backslash \overline{\{x_0 + \epsilon\sigma\}}$, with $x_0 = 0$ i.e. $\Omega_\epsilon = \Omega \backslash \overline{\{\epsilon\sigma\}}$ where σ is the straight crack $\sigma = \{(s, 0), -1 < s < 1\}$. We denote $\sigma_\epsilon = \epsilon\sigma$. For a detailed version of the calculus, we refer the reader to [6]. We have to estimate the difference $J_\epsilon(u_\epsilon) - J_0(u_0)$. By using Eqs. (5) and (6) for u_ϵ and u_0, we get:

$$J_\epsilon(u_\epsilon) - J_0(u_0) = L_\epsilon(u_\epsilon - u_0) + \mathcal{I}_\epsilon$$

with

$$L_\epsilon(v) = \int_{\Omega_\epsilon} (f - 2u_0) v \, dx \quad \text{and} \quad \mathcal{I}_\epsilon = -\int_{\Omega_\epsilon} (u_\epsilon - u_0)^2 \, dx \tag{7}$$

The first key point is to introduce $v_\epsilon \in H(\Omega_\epsilon)$ as the unique solution of the dual problem (see [9] and [2]):

$$a_\epsilon(u, v_\epsilon) = -L_\epsilon(u), \quad \forall u \in H(\Omega_\epsilon) \tag{8}$$

By extending σ to a closed curve, by using integration by parts and the equation verified by u_0, we obtain

$$J_\epsilon(u_\epsilon) - J_0(u_0) = \int_{\sigma_\epsilon} \left(B_1(u_0) - \alpha \partial_n u_0\right)[v_\epsilon] - B_2(u_0)[\partial_n v_\epsilon] + \mathcal{I}_\epsilon \tag{9}$$

where $[v_\epsilon]$ and $[\partial_n v_\epsilon]$ denote respectively the jump of v_ϵ and $\partial_n v_\epsilon$ across σ_ϵ. Then we set $w_\epsilon = v_\epsilon - v_0$ where v_0 is the solution of (8) with $\epsilon = 0$ ($\Omega_0 = \Omega$), thus:

$$J_\epsilon(u_\epsilon) - J_0(u_0) = \int_{\sigma_\epsilon} \left(B_1(u_0) - \alpha \partial_n u_0\right)[w_\epsilon] - B_2(u_0)[\partial_n w_\epsilon] + \mathcal{I}_\epsilon = I_{\mathrm{A}} - I_{\mathrm{B}} + \mathcal{I}_\epsilon \tag{10}$$

with

$$I_{\mathrm{A}} = \int_{\sigma_\epsilon} \left(B_1(u_0) - \alpha \partial_n u_0\right)[w_\epsilon], \qquad I_{\mathrm{B}} = \int_{\sigma_\epsilon} B_2(u_0)[\partial_n w_\epsilon] \tag{11}$$

The estimation of the two terms of (11) needs to approximate w_ϵ. It is classical (see [2]) to approximate w_ϵ by the solution of the exterior problem

$$\begin{cases} \Delta^2 P = 0, & \text{in } \mathbb{R}^2 \backslash \overline{\sigma} \\ B_1(P) = g_1, \quad B_2(P) = g_2 & \text{on } \sigma \end{cases} \tag{12}$$

where g_1 end g_2 are known functions defined from second derivatives of v_0 at point 0. The solution of (12) is the sum of triple and quadruple layer potentials (see [6] for more details). We can show that $w_\epsilon = v_\epsilon - v_0$ can be written as $w_\epsilon(x) = \epsilon^2 P(\frac{x}{\epsilon}) + e_\epsilon$ with $\|e_\epsilon\|_{2,\Omega_\epsilon} = 0(-\epsilon^2 \log(\epsilon))$ and then the three terms in (10) have the following expansion:

$$I_{\mathrm{A}} = o\left(\epsilon^2\right), \qquad I_{\mathrm{B}} = \epsilon^2 \gamma \int_\sigma \lambda_2(y) \, d\sigma_y + o\left(\epsilon^2\right), \qquad \mathcal{I}_\epsilon = o\left(\epsilon^2\right) \tag{13}$$

with $\gamma = \frac{\partial^2 u_0}{\partial x_2^2}(0) + \nu \frac{\partial^2 u_0}{\partial x_1^2}(0)$ and $\lambda_2(s) = \frac{-4\beta}{(1-\nu)(3+\nu)}\sqrt{1-s^2}, \forall s \in]-1,1[$, where $\beta = \frac{\partial^2 v_0}{\partial x_2^2}(0) + \nu \frac{\partial^2 v_0}{\partial x_1^2}(0)$. Gathering estimates (10) and (13), we get the expression of the topological gradient for the case $\Omega_\epsilon = \Omega \backslash \overline{\sigma_\epsilon}$ and $x_0 = 0$:

$$\mathcal{I}(0) = -\frac{2\pi}{(1-\nu)(3+\nu)}\left(\frac{\partial^2 u_0}{\partial x_2^2}(0) + \nu \frac{\partial^2 u_0}{\partial x_1^2}(0)\right)\left(\frac{\partial^2 v_0}{\partial x_2^2}(0) + \nu \frac{\partial^2 v_0}{\partial x_1^2}(0)\right) \tag{14}$$

Remark 2. Thanks to a basis change, the topological gradient for a straight crack centered at $x_0 \in \Omega$ and of normal n is given by

$$\mathcal{I}(x_0, n) = \frac{-2\pi}{(1-\nu)(3+\nu)}\left(\nabla^2 u_0(x_0)(n,n) + \nu \nabla^2 u_0(x_0)(\tau,\tau)\right)\left(\nabla^2 v_0(x_0)(n,n) + \nu \nabla^2 v_0(x_0)(\tau,\tau)\right) \tag{15}$$

with τ the tangent vector such as (τ, n) is an orthonormal basis of $\mathbb{R}^2$ and the topological gradient at point $x_0 \in \Omega$ is defined as:

$$\mathcal{I}(x_0) = \min_{\|n\|=1} \mathcal{I}(x_0, n) \tag{16}$$

Remark 3. In the case of the ball: $\Omega_\epsilon = \Omega \backslash \overline{x_0 + \epsilon B(0,1)}$, the computation is identical (see [6] and [1]):

$$\begin{aligned}\mathcal{I}(x_0) = {} & \pi\left(f(x_0) - u_0(x_0)\right)\left(v_0(x_0) - u_0(x_0)\right) - \frac{\pi(1+\nu)}{1-\nu}\Delta u_0(x_0)\Delta v_0(x_0) \\ & - \frac{2\pi(1-\nu)}{3+\nu}\left(\left(\frac{\partial^2 u_0}{\partial x_1^2}(x_0) - \frac{\partial^2 u_0}{\partial x_2^2}(x_0)\right)\left(\frac{\partial^2 v_0}{\partial x_1^2}(x_0) - \frac{\partial^2 v_0}{\partial x_2^2}(x_0)\right) + 4\frac{\partial^2 u_0}{\partial x_2 \partial x_1}(x_0)\frac{\partial^2 v_0}{\partial x_2 \partial x_1}(x_0)\right)\end{aligned} \tag{17}$$

4. Numerical approximation and experimental results

The computation of the topological gradient only need to compute the solutions of the two following variational problems:

$$\begin{aligned} & a_0(u_0, v) = l_0(v), \quad \forall v \in H(\Omega) \\ & a_0(u, v_0) = -L_0(u), \quad \forall u \in H(\Omega) \end{aligned} \tag{18}$$

where l_0 and L_0 are respectively defined by (4) and (7) with $\epsilon = 0$. To discretize (18), we used Morley finite elements (see [4] and [8]). If $(\mathcal{T}_h)_h$ is a regular triangulation of Ω, let X_h be the Morley finite elements space associated with $(\mathcal{T}_h)_h$. We define on X_h the bilinear and linear forms:

$$a_{0,h}(u,v) = \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{K}} \sum_{i,j=1}^{2} \frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial^2 v}{\partial x_i \partial x_j}\,\mathrm{d}x$$

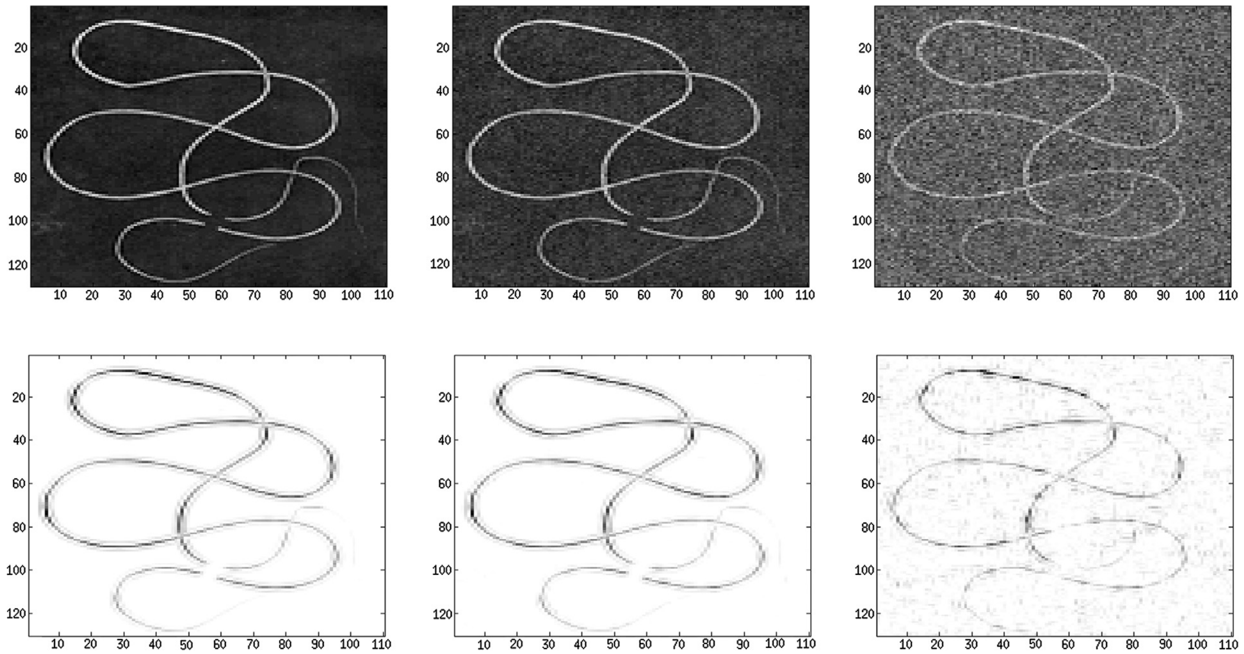
$$l_{0,h} = \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{K}} f v \,\mathrm{d}x$$

$$L_{0,h}(v) = \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{K}} (f - 2u_{0,h}) v \,\mathrm{d}x$$

We denote respectively by $u_{0,h}$ and $v_{0,h}$ the solution of the problems:

$$\begin{aligned} & a_{0,h}(u_{0,h}, v_h) = l_{0,h}(v_h), \quad \forall v_h \in X_h \\ & a_{0,h}(u_h, v_{0,h}) = -L_{0,h}(u_h), \quad \forall u_h \in X_h \end{aligned}$$

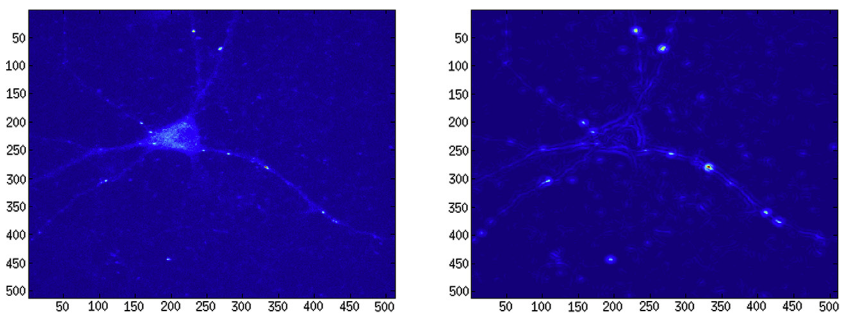
Then the discrete topological gradient is computed thanks to formula (16) evaluated at vertices of $(\mathcal{T}_h)_h$. We illustrate our approach by giving two experimental results in Fig. 1 and Fig. 2. To determine the size of the structures to be detected, we add in front of the bi-Laplacian operator a regularizing parameter $\alpha > 0$ to be tuned by the user, for example for structures of size 1 to 6 pixels $\alpha = 0.1$ and for structures of size 6 to 17 pixels $\alpha = 3$. Note also that our topological gradient (TG) is able to distinguish between fine structures and jumps (edges). In the neighborhood of an edge, the TG is not equal to 0 but it is weak. On the other hand, in the neighborhood of a fine structure it is big, so with a correct threshold we can easily detect fine structures and eliminate jumps.



(a) (b) (c)

Fig. 1. Robustness of the topological gradient with respect to noise on a synthesis image using (16). (a) Image without noise. (b) Noisy image with Gaussian noise, PSNR = 26 dB. (c) Noisy image with Gaussian noise, PSNR = 14 dB. Top: initial image, down: topological gradient using (16).

Fig. 1. Robustesse du gradient topologique par rapport au bruit pour une image synthétique en utilisant (16) : (a) une image sans bruit, (b) une image bruitée par du bruit gaussien, PSNR = 26 dB, (c) image bruitée par du bruit gaussien, PSNR = 14 dB. En haut : image initiale, en bas : gradient topologique utilisant (16).



(a) Initial image, image initiale

(b) Topological gradient, gradient topologique

Fig. 2. Topological gradient on a real biological image (rat's neuron) using (16).

Fig. 2. Gradient topologique sur une image réelle biologique (neurone de rat) utilisant (16).

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