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Differential geometry

On anti-Hermitian metric connections ☆

Sur les connexions métriques anti-hermitiennes

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ABSTRACT

It is a remarkable fact that anti-Kähler and its twin metrics share the same Levi–Civita connection. Such torsion-free metric connection also emphasizes the importance of anti-Hermitian metric connections with torsion in the study of anti-Hermitian geometry. With the objective of defining new types of anti-Hermitian metric connections, in the present paper we consider classes of anti-Hermitian manifolds associated with these connections.

RÉSUMÉ

C'est un fait remarquable que les métriques anti-kählériennes et leurs métriques jumelles possèdent la même connexion de Levi–Civita. De telles connexions métriques sans torsion mettent aussi en relief l'importance des connexions métriques anti-hermitiennes avec torsion dans l'étude de la géométrie anti-hermitienne. Dans le but de définir de nouveaux types de connexions métriques anti-hermitiennes, nous considérons dans la présente note des classes de variétés anti-hermitiennes associées à ces connexions.

1. Introduction

Let (M, J) be a $2n$-dimensional almost complex manifold, where J denotes its almost complex structure. A semi-Riemannian metric g of neutral signature (n, n) is an anti-Hermitian (also known as a Norden) metric if $g(JX, Y) = g(X, JY)$ for any $X, Y \in \aleph(M)$, where $\aleph(M)$ is the module of vector fields on M. An almost complex manifold (M, J) with an anti-Hermitian metric is referred to as an almost anti-Hermitian manifold. An anti-Kähler (Kähler–Norden) manifold can be defined as a triple (M, g, J) which consists of a smooth manifold M endowed with an almost complex structure J and an anti-Hermitian metric g such that $\nabla J = 0$, where ∇ is the Levi–Civita connection of g. It is well known that the condition $\nabla J = 0$ is equivalent to C-holomorphicity (analyticity) of the anti-Hermitian metric g [2], i.e. $\Phi_J g = 0$, where $(\Phi_J g)(X, Y, Z) = (L_{JX} g - L_X G)(Y, Z) = -(\nabla_X G)(Y, Z) + (\nabla_Y G)(Z, X) + (\nabla_Z G)(X, Y)$ is the Tachibana operator [4,6], $G(Y, Z) = (g \circ J)(Y, Z) = g(JY, Z)$ is the twin anti-Hermitian metric. Since in dimension 2 an anti-Kähler manifold is flat, we assume in the sequel that $\dim M \geq 4$.

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It is well known that the pair (J, g) of an almost Hermitian structure defines a fundamental 2-form Ω by $\Omega(X, Y) = g(JX, Y)$. If the skew-symmetric tensor Ω is a Killing-Yano tensor, i.e.

$$(\nabla_X \Omega)(Y, Z) + (\nabla_Y \Omega)(X, Z) = 0 \tag{1}$$

or equivalently if the almost complex structure J satisfies $(\nabla_X J)Y + (\nabla_Y J)X = 0$ for any $X, Y \in \aleph(M)$, then the manifold is called a nearly Kähler manifold (also known as K-spaces or almost Tachibana spaces).

Let now (M, g, J) be an almost anti-Hermitian manifold. Then the pair (J, g) defines, as usual, the twin anti-Hermitian metric $G(Y, Z) = (g \circ J)(Y, Z) = g(JY, Z)$, but G is symmetric, rather than a 2-form Ω. Thus, the anti-Hermitian pair (J, g) does not give rise to a 2-form, and the Killing–Yano equation (1) has no immediate meaning. Therefore we can replace the Killing–Yano equation by the Codazzi equation

$$(\nabla_X G)(Y, Z) - (\nabla_Y G)(X, Z) = 0. \tag{2}$$

Eq. (2) is equivalent to

$$(\nabla_X J)Y - (\nabla_Y J)X = 0. \tag{3}$$

If the almost complex structure of almost anti-Hermitian manifold satisfies (3), then the triple (M, J, g) is called an anti-Kähler–Codazzi manifold [5].

Let the tensor G (i.e. the twin anti-Hermitian metric) be a Killing symmetric tensor, i.e. $\sigma_{X,Y,Z}(\nabla_X G)(Y, Z) = 0$, where σ is the cyclic sum with respect to X, Y and Z. This is the class of the quasi-Kähler manifold with anti-Hermitian (Norden) metric [3].

By far the most interesting integrable manifolds are the anti-Kähler–Codazzi manifolds. The almost complex structure J is integrable if $N_J = 0$, where N_J is the Nijenhuis tensor of J or alternatively, if there exists a torsion-free connection $\overset{*}{\nabla}$ ($\overset{*}{\nabla} \neq \nabla$ in the general case) such that $\overset{*}{\nabla} J = 0$. We observe in [5] that anti-Kähler–Codazzi manifolds are integrable almost anti-Hermitian manifolds.

2. Anti-Hermitian metric connections

In [2,4,5], we have given the anti-Hermitian metric g and considered exclusively the Levi–Civita connection ∇ of g. This is the unique connection that satisfies $\nabla g = 0$, and has no torsion. But there are many other connections $\tilde{\nabla}$ with torsion parallelizing the metric g. We call these connections anti-Hermitian metric connections.

Let (M, g, J) be an almost anti-Hermitian manifold. If we introduce a connection $\tilde{\nabla}$ and put $\tilde{\nabla}_X Y = \nabla_X Y + S(X, Y)$ for any $X, Y \in \aleph(M)$, where ∇ is the Levi–Civita connection of g, then S is a tensor field of type $(1, 2)$ and the torsion tensor T of connection $\tilde{\nabla}$ is given by

$$\begin{aligned} T(X, Y) &= \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] = \nabla_X Y + S(X, Y) - \nabla_Y X - S(Y, X) - [X, Y] \\ &= T_\nabla(X, Y) + S(X, Y) - S(Y, X) = S(X, Y) - S(Y, X). \end{aligned} \tag{4}$$

For the covariant derivative $\tilde{\nabla}$ of g, we have:

$$\begin{aligned} (\tilde{\nabla}_X g)(Y, Z) &= X\big(g(Y, Z)\big) - g(\tilde{\nabla}_X Y, Z) - g(Y, \tilde{\nabla}_X Z) \\ &= X\big(g(Y, Z)\big) - g\big(\nabla_X Y + S(X, Y), Z\big) - g\big(Y, \nabla_X Z + S(X, Z)\big) \\ &= (\nabla_X g)(Y, Z) - g\big(S(X, Y), Z\big) - g\big(Y, S(X, Z)\big) \\ &= -g\big(S(X, Y), Z\big) - g\big(Y, S(X, Z)\big). \end{aligned}$$

Consequently, in order to have $\tilde{\nabla} g = 0$, it is necessary and sufficient that

$$g\big(S(X, Y), Z\big) + g\big(Y, S(X, Z)\big) = 0.$$

From here, we have the following theorem.

Theorem 2.1. *Let (M, g, J) be an almost anti-Hermitian manifold. A connection $\tilde{\nabla} = \nabla + S$ is metric connection of g (i.e. $\tilde{\nabla} g = 0$) if and only if*

$$S(X, Y, Z) + S(X, Z, Y) = 0, \tag{5}$$

where $S(X, Y, Z) = g(S(X, Y), Z)$.

Now putting $T(X,Y,Z) = g(T(X,Y),Z)$, from (4) we have:

$$T(X,Y,Z) = S(X,Y,Z) - S(Y,X,Z).$$

Similarly,

$$T(Z,X,Y) = S(Z,X,Y) - S(X,Z,Y),$$
$$T(Z,Y,X) = S(Z,Y,X) - S(Y,Z,X).$$

Using (5), from these three equations we obtain:

$$S(X,Y,Z) = \frac{1}{2}\big(T(X,Y,Z) + T(Z,X,Y) + T(Z,Y,X)\big).$$

3. Metric connections of twin anti-Hermitian metrics

Let now $G(Y,Z) = (g \circ J)(Y,Z) = g(JY,Z)$ be a twin anti-Hermitian metric. Then, in order that to have $\tilde{\nabla}G = 0$, it is necessary and sufficient that we have

$$\begin{aligned}(\tilde{\nabla}_X G)(Y,Z) &= (\nabla_X G)(Y,Z) - G\big(S(X,Y),Z\big) - G\big(Y,S(X,Z)\big) \\ &= (\nabla_X G)(Y,Z) - g\big(JS(X,Y),Z\big) - g\big(JY,S(X,Z)\big) \\ &= (\nabla_X G)(Y,Z) - g\big(S(X,Y),JZ\big) - g\big(S(X,Z),JY\big) \\ &= (\nabla_X G)(Y,Z) - S(X,Y,JZ) - S(X,Z,JY) = 0,\end{aligned}$$

which is equivalent to

$$(\nabla_X G)(Y,Z) - S'_J(X,Y,Z) - S'_J(X,Z,Y) = 0, \tag{6}$$

where $S'_J(X,Y,Z) = S(X,Y,JZ)$.

The connection $\tilde{\nabla}$ is not completely determined by (5) and (6). So we can introduce some other condition on S. We try to solve the equation with respect to S. From now on, we assume only $\tilde{\nabla}G = 0$ and make no use of $\tilde{\nabla}g = 0$ (i.e. $S(X,Y,Z) + S(X,Z,Y) = 0$). This latter equation will be satisfied in special cases as a consequence of the equation introduced in next sections.

4. Anti-Hermitian metric connection of type I

The metric connection $\tilde{\nabla} = \nabla + S$ of g ($\tilde{\nabla}g = 0$) is called an anti-Hermitian metric connection of type I if $\tilde{\nabla}G = 0$ and

$$S'_J(X,Y,Z) - S'_J(X,Z,Y) = 0. \tag{7}$$

From (6) and (7), we have

$$S'_J(X,Y,Z) = \frac{1}{2}(\nabla_X G)(Y,Z)$$

from which

$$S(X,Y,JZ) = \frac{1}{2}g\big((\nabla_X J)Y,Z\big),$$
$$g\big(S(X,Y),JZ\big) = \frac{1}{2}g\big((\nabla_X J)Y,Z\big),$$
$$g\big(JS(X,Y),Z\big) = \frac{1}{2}g\big((\nabla_X J)Y,Z\big),$$
$$JS(X,Y) = \frac{1}{2}(\nabla_X J)Y \tag{8}$$

or

$$S(X,Y) = \frac{1}{2}J(\nabla_X J)Y. \tag{9}$$

If we substitute JZ into Z in the second equation of (8), we have:

$$S(X,Y,Z) = -\frac{1}{2}g\big((\nabla_X J)Y,JZ\big). \tag{10}$$

On the other hand, using

$$g\big((\nabla_X J)Z, JY\big) = g\big(Z, (\nabla_X J)JY\big) = -g\big(Z, J\big((\nabla_X J)Y\big)\big) = -g\big((\nabla_X J)Y, JZ\big)$$

we have

$$S(X, Z, Y) + S(X, Y, Z) = -\frac{1}{2}\big(g\big((\nabla_X J)Y, JZ\big) + g\big((\nabla_X J)Z, JY\big)\big) = 0.$$

Thus, in an anti-Hermitian manifold, the tensor S in the form (10) satisfies Eq. (5) (i.e. $\tilde{\nabla} g = 0$) and consequently the connection $\tilde{\nabla} = \nabla + \frac{1}{2} J(\nabla J)$ is anti-Hermitian metric connection of type I.

From (4) and (9), we see that the torsion tensor of the connection $\tilde{\nabla} = \nabla + S$ is given by:

$$T(X, Y) = -\frac{1}{2} J\big((\nabla_X J)Y - (\nabla_Y J)X\big). \tag{11}$$

Let now the triple (M, g, J) be an anti-Kähler–Codazzi manifold. Then from (3) and (11) we find that $T = 0$, i.e. in an anti-Kähler–Codazzi manifold the anti-Hermitian metric connection of type I reduces to a Levi–Civita connection. Thus we have the following theorem.

Theorem 4.1. *Every anti-Hermtian manifold (M, g, J) admits an anti-Hermitian metric connection of type I. If an anti-Hermitian manifold is anti-Kähler–Codazzi, then the anti-Hermitian metric connection of type I coincides with the Levi–Civita connection of g, i.e. g and G (twin metric) share the same Levi–Civita connection.*

5. Anti-Hermitian metric connection of type II

The metric connection $\tilde{\nabla} = \nabla + S$ of g ($\tilde{\nabla} g = 0$) is called an anti-Hermitian metric connection of type II if $\tilde{\nabla} G = 0$ and

$$S'_J(X, Y, Z) - S'_J(Z, Y, X) = 0. \tag{12}$$

From (6), we have

$$(\nabla_X G)(Y, Z) - S'_J(X, Y, Z) - S'_J(X, Z, Y) = 0,$$
$$(\nabla_Y G)(Z, X) - S'_J(Y, Z, X) - S'_J(Y, X, Z) = 0,$$
$$(\nabla_Z G)(X, Y) - S'_J(Z, X, Y) - S'_J(Z, Y, X) = 0,$$

and consequently, taking account of (12), we find

$$\begin{aligned}&(\nabla_X G)(Y, Z) - (\nabla_Y G)(Z, X) + (\nabla_Z G)(X, Y)\\ &\quad = 2S'_J(X, Y, Z) = 2S(X, Y, JZ) = 2g\big(S(X, Y), JZ\big) = 2g\big(JS(X, Y), Z\big).\end{aligned} \tag{13}$$

Since in an anti-Hermitian manifold the operator $\Phi_J g$ reduces to form (see [2,4])

$$(\Phi_J g)(Y, Z, X) = (\nabla_X G)(Y, Z) - (\nabla_Y G)(Z, X) + (\nabla_Z G)(X, Y)$$

and the anti-Kähler condition ($\nabla J = 0$) is equivalent to $(\Phi_J g)(Y, Z, X) = 0$, from (13) we have $S = 0$ and the metric connection $\tilde{\nabla}$ reduces to a Levi–Civita connection ∇ and it is clear that the tensor $S = 0$ satisfies Eq. (5) and consequently the connection $\tilde{\nabla} = \nabla$ is anti-Hermitian metric connection. Thus we have

Theorem 5.1. *If an anti-Hermitian manifold is anti-Kähler, then the anti-Hermitian metric connection $\tilde{\nabla}$ of type II coincides with the Levi–Civita connection ∇.*

Let now (M, g, J) be an anti-Kähler–Codazzi manifold. Since $(\nabla_Z G)(X, Y) = (\nabla_Z G)(Y, X)$, from (3) and (13) we find

$$2g\big(JS(X, Y), Z\big) = (\nabla_X G)(Y, Z) - (\nabla_Y G)(Z, X) + (\nabla_Z G)(Y, X) = (\nabla_X G)(Y, Z) = g\big((\nabla_X J)Y, Z\big)$$

or

$$S(X, Y) = -\frac{1}{2} J(\nabla_X J)Y. \tag{14}$$

By similar devices as above, we easily see that the tensor S in the form (14) satisfies Eq. (5) and consequently the connection $\tilde{\nabla} = \nabla - \frac{1}{2} J(\nabla J)$ is an anti-Hermitian metric connection of type II. Thus we have the following theorem.

Theorem 5.2. *If an anti-Hermitian manifold is anti-Kähler–Codazzi, then the anti-Hermitian metric connection* $\tilde{\nabla}$ *of type II is given by* $\tilde{\nabla} = \nabla - \frac{1}{2}J(\nabla J)$.

Similarly, if (M, g, J) is quasi-Kähler, i.e. $(\nabla_X G)(Y, Z) + (\nabla_Y G)(Z, X) + (\nabla_Z G)(X, Y) = 0$, then from (13) we find

$$S(X, Y) = J(\nabla_X J)Y$$

which satisfies (5). Thus we have the following theorem.

Theorem 5.3. *If an anti-Hermitian manifold is quasi-Kähler, then the anti-Hermitian metric connection* $\tilde{\nabla}$ *of type II is given by* $\tilde{\nabla} = \nabla + J(\nabla J)$.

Remark 1. Given a Hermitian manifold (M, g, J), there is a unique connection $\tilde{\nabla}$ (known as the Bismut connection [1]) with totally skew torsion which preserves both the complex structure and the Hermitian metric, i.e., $\tilde{\nabla} g = 0$ and $\tilde{\nabla} J = 0$. For the anti-Hermitian manifolds, from $\tilde{\nabla} g = 0$ and $\tilde{\nabla} G = 0$ we have $\tilde{\nabla} J = 0$, therefore in some aspects anti-Hermitian metric connections of types I and II introduced in the present paper are similar to the Bismut connection.

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