C. R. Acad. Sci. Paris, Ser. I 352 (2014) 715–718

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Geometry/Differential geometry

Blowing-up points on locally conformally balanced manifolds

Éclatement de points dans les variétés localement conformément équilibrées

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ARTICLE INFO

Article history:
Received 24 March 2014
Accepted after revision 11 July 2014
Available online 1 August 2014

Presented by Claire Voisin

ABSTRACT

In this note, we show that the blowing-up of a point on a locally conformally balanced manifold also admits a locally conformally Balanced manifold structure.

RÉSUMÉ

Dans cette note, nous montrons que l'éclatement d'un point dans une variété localement conformément équilibrée admet aussi une structure de variété localement conformément équilibrée.

1. Introduction

Let (M, J, g) be an n-dimensional complex Hermitian manifold and let ω be its Kähler form ($n \geq 3$). If $d\omega^{n-1} = 0$, then ω is called a Balanced metric. A complex manifold is endowed with a Balanced metric is called a Balanced manifold.

An n-dimensional complex Hermitian manifold (M, J, g) is called a *locally conformally balanced manifold* if there exists an open covering $\{U_i\}$ and a family of smooth real valued functions $f_i : U_i \to \mathbb{R}$ such that for each local metric $\hat{g}_i := e^{-f_i} g|_{U_i}$ is a balanced metric on U_i. If there is a globally defined smooth real valued function $f : X \to \mathbb{R}$ such that the metric $e^{-f} g$ is balanced, then (M, J) is called a *globally conformally balanced manifold*.

It is well known that the blowing-up of points on a Kähler manifold (cf. Voisin [5]), a balanced manifold (cf. Michelsohn [3]) and a locally conformally Kähler manifold (cf. Tricerri [4] or Vuletescu [6]) also admits Kähler metric, balanced metric and locally conformally Kähler metric, respectively.

In this note, we will consider the blowing-up of point on locally conformally balanced manifold and show the following result.

Theorem 1.1. *Assume that (M, J, g) is a locally conformally balanced manifold. Then, the blowing-up $\hat{M}$, of M at any point, also admits a locally conformally balanced metric.*

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http://dx.doi.org/10.1016/j.crma.2014.07.005

Remark 1.2. This result is inspired by Theorem 1 of Vuletescu [6].

Next, to prove our main result, we will give an equivalent definition of locally conformally balanced manifolds. First, we recall that the *Lee form* of an n-dimensional complex Hermitian manifold (M, J, g) is the 1-form:

$$\theta := \frac{1}{n-1} J\mathrm{d}^*\omega,$$

where d^* denotes the coderivative.

According to the formulas $*\omega = \frac{1}{(n-1)!}\omega^{n-1}$ and $-*J\theta = \frac{1}{(n-1)!}\theta \wedge \omega^{n-1}$, it is easy to see that

$$\mathrm{d}\omega^{n-1} = (n-1)\theta \wedge \omega^{n-1} \quad \Leftrightarrow \quad \theta = \frac{1}{n-1} J\mathrm{d}^*\omega. \tag{1}$$

Proposition 1.3. *An n-dimensional complex Hermitian manifold (M, J, g) is a locally conformally balanced manifold if and only if there exists a closed 1-form satisfying Eq. (1).*

Proof. "⇒:" If the Hermitian manifold (M, J, g) is a locally conformally Balanced manifold, then there exists an open covering $\{U_i\}$ of complex manifold (M, J) and a family of smooth real valued functions $f_i : U_i \to \mathbb{R}$ such that

$$\hat{g}_i := \mathrm{e}^{-f_i} g|_{U_i},$$

is a balanced metric, its Kähler form $\hat{\omega} := \mathrm{e}^{-f_i}\omega$. Then,

$$\begin{aligned} 0 = \mathrm{d}\hat{\omega}^{n-1} &= \mathrm{d}\big(\mathrm{e}^{-(n-1)f_i}\omega^{n-1}\big) \\ &= -(n-1)\mathrm{e}^{-(n-1)f_i}\mathrm{d}f_i \wedge \omega^{n-1} + \mathrm{e}^{-(n-1)f_i}\mathrm{d}\omega^{n-1} \\ &= \mathrm{e}^{-(n-1)f_i}\big(\mathrm{d}\omega^{n-1} - (n-1)\mathrm{d}f_i \wedge \omega^{n-1}\big), \end{aligned}$$

on U_i. This implies that

$$\mathrm{d}\omega^{n-1} = (n-1)\mathrm{d}f_i \wedge \omega^{n-1}.$$

Then we get $\mathrm{d}f_i = \frac{1}{n-1} J\mathrm{d}^*\omega|_{U_i}$. Hence we get a globally defined closed 1-form $\theta := \{\mathrm{d}f_i, U_i\}$ satisfying Eq. (1).

"⇐:" If there is a closed 1-form θ, such that

$$\mathrm{d}\omega^{n-1} = (n-1)\theta \wedge \omega^{n-1}.$$

Because θ is closed, the Poincaré lemma implies that θ is locally exact form. This means that we obtain an open covering $\{U_i\}$ of complex manifold (M, J) and a family of smooth real valued functions $f_i : U_i \to \mathbb{R}$ such that $\theta|_{U_i} = \mathrm{d}f_i$ for every i. Then we have

$$\mathrm{d}\omega^{n-1} = (n-1)\mathrm{d}f_i \wedge \omega^{n-1},$$

on U_i. This implies that

$$\begin{aligned} \mathrm{d}\big(\mathrm{e}^{-(n-1)f_i}\omega^{n-1}\big) &= -(n-1)\mathrm{d}f_i\mathrm{e}^{-(n-1)f_i} \wedge \omega^{n-1} + \mathrm{e}^{-(n-1)f_i}\mathrm{d}\omega^{n-1} \\ &= (n-1)\mathrm{e}^{-(n-1)f_i}(-\mathrm{d}f_i + \mathrm{d}f_i) \wedge \mathrm{d}\omega^{n-1} \\ &= 0, \end{aligned}$$

on U_i. □

Now, we may give the following equivalent definition of locally conformally balanced manifolds.

Definition 1.4. We say that an n-dimensional complex Hermitian manifold (M, J, g) is called a *locally conformal balanced manifold* if there exists a globally defined closed 1-form θ such that

$$\mathrm{d}\omega^{n-1} = (n-1)\theta \wedge \omega^{n-1}.$$

If the 1-form θ is exact, then the Hermitian manifold (M, J, g) is called a *globally conformal balanced manifold*.

Here, we give a class of examples of locally conformally balanced manifolds. It is still very interesting to construct more non-trivial high-dimensional locally conformally balanced manifolds.

Example 1.5. In [1], Fino and Tomassini show that there exists a non-trivial compact $\mathbb{T}^2$-bundle, over any non-Kähler compact homogeneous complex surface, which carries a locally conformally balanced structure. Indeed, this $\mathbb{T}^2$-bundle carries much more structures; for more details refer to [1, Theorem 5.1].

2. Proof of Theorem 1.1

Suppose that $\pi : \hat{M} \to M$ is a blowing up of M along a point p. We denote by $E := \pi^{-1}(p)$ the exceptional divisor of the blowing up. Let ω be the Kähler form of the locally conformally balanced metric on the locally conformally Balanced manifold (M, J, g, θ). Then the pullback form $\pi^*\omega$ is a $(1,1)$-form on $\hat{M}$, which is strictly positive definite on $\hat{M}\backslash E$, such that

$$d\big(\pi^*\omega\big)^{n-1} = (n-1)\pi^*(\theta) \wedge \big(\pi^*\omega\big)^{n-1}.$$

Since E is simply connected, according to [4, Lemma 4.4] there exists an open neighborhood U of E in $\hat{M}$ and a smooth function $f : \hat{M} \to \mathbb{R}$ such that $\tilde{\omega} := e^f \pi^*\omega$ satisfying

$$\begin{aligned} d\tilde{\omega}^{n-1} &= de^{(n-1)f}\big(\pi^*\omega\big)^{n-1} \\ &= (n-1)e^{(n-1)f} df \wedge \big(\pi^*\omega\big)^{n-1} + e^{(n-1)f} d\big(\pi^*\omega\big)^{n-1} \\ &= (n-1)e^{(n-1)f} df \wedge \big(\pi^*\omega\big)^{n-1} + (n-1)e^{(n-1)f}\pi^*\theta \wedge \big(\pi^*\omega\big)^{n-1} \\ &= (n-1)\big(df + \pi^*\theta\big) \wedge \tilde{\omega}^{n-1}, \end{aligned}$$

and such that the 1-form $\tilde{\theta} := df + \pi^*\theta$ satisfies $\tilde{\theta}|_U = 0$. Because of θ is closed, we have $d\tilde{\theta} = ddf + d\pi^*\theta = 0$.

One can find a Hermitian metric in the holomorphic line bundle $\mathcal{O}_{\hat{M}}(E)$ on $\hat{M}$ associated with the exceptional divisor E, such that the curvature Ω_E ($d\Omega_E = 0$) of its canonical connection satisfies the following conditions (for details, please refer to Griffiths–Harris [2, pp. 185–187]):

(i) Ω_E is strictly negative definite along E, i.e. $\Omega_E(u, \bar{u}) < 0$ for every non-vanishing vector $u \in T_{\hat{x}}(E)$ and for every $\hat{x} \in E$;
(ii) Ω_E is negatively semi-definite at points of E, i.e. $\Omega_E(u, \bar{u}) \le 0$ for any $\hat{x} \in E$ and any $u \in T_{\hat{x}}(\hat{M})$;
(iii) and zero outside U.

We set $\hat{\Omega} := N\tilde{\omega}^{n-1} + (-\Omega_E)^{n-1}$, which is a real $(n-1, n-1)$-form, here N is a positive integer. For some larger positive N, we will show that $\hat{\Omega}$ is strictly positive definite $(n-1, n-1)$-form as follows.

In fact, since Ω_E is vanishing outside of U, hence $\hat{\Omega}$ is a strictly positive definite $(n-1, n-1)$-form outside of U as $N\tilde{\omega}^{n-1}$ is a strictly positive definite $(n-1, n-1)$-form for any $N > 0$. As we know, since both $\tilde{\omega}$ and $-\Omega_E$ are positive semi-definite at the exceptional divisor E, so $\tilde{\omega}^{n-1}$ and $(-\Omega_E)^{n-1}$ are also positive semi-definite at E, then we only need to show the definiteness of $\hat{\Omega}$ at the points of E. Given a point $\hat{x} \in E$ and any vectors $u_1, u_2, \ldots, u_{n-1} \in T_{\hat{x}}\hat{M}$. We assume that $\hat{\Omega}(u_1, \bar{u}_1, u_2, \bar{u}_2, \ldots, u_{n-1}, \bar{u}_{n-1}) = 0$. Since $\tilde{\omega}^{n-1}$ and $(-\Omega_E)^{n-1}$ are positive semi-definite, we have:

$$\tilde{\omega}^{n-1}(u_1, \bar{u}_1, \ldots, u_{n-1}, \bar{u}_{n-1}) = 0, \tag{2}$$

$$(-\Omega_E)^{n-1}(u_1, \bar{u}_1, \ldots, u_{n-1}, \bar{u}_{n-1}) = 0. \tag{3}$$

By $\tilde{\omega}^{n-1} = \pi^*\omega^{n-1}$ and Eq. (2), we have:

$$\omega^{n-1}(\pi_{*,\hat{x}}u_1, \pi_{*,\hat{x}}\bar{u}_1, \ldots, \pi_{*,\hat{x}}u_{n-1}, \pi_{*,\hat{x}}\bar{u}_{n-1}) = 0.$$

Because ω^{n-1} is strictly positive definite, hence $u_i \in \mathrm{Ker}(\pi_{*,\hat{x}})$. Since $\mathrm{Ker}(\pi_{*,\hat{x}}) = T_{\hat{x}}E$, so $u_i \in T_{\hat{x}}E$ for all $i = 1, \ldots, n-1$. As $-\Omega_E$ is strictly positive definite along E, so $(-\Omega_E)^{n-1}$ is strictly positive definite along E. This contradicts Eq. (3), thus $u_i = 0$ for all $i = 1, \ldots, n-1$. This shows that $\tilde{\omega}^{n-1}$ and $(-\Omega_E)^{n-1}$ are strictly positive definite at $\hat{x} \in E$.

To prove the definiteness of $\hat{\Omega}$ on U, it is sufficient to see that, for any point $\hat{x} \in U$, there exists some positive integer $N_{\hat{x}}$ such that $N\tilde{\omega}^{n-1} + (-\Omega_E)^{n-1}$ is strictly positive definite at $\hat{x}$ for any $N > N_{\hat{x}}$, hence it is strictly positive definite on a neighborhood $U_{\hat{x}}$ of $\hat{x}$. Thanks to the fact that U is relatively compact, we can cover U by finitely such $U_{\hat{x}}$, and denote by $N_{\max}$ the maximum of $N_{\hat{x}}$.

This show that the real $(n-1, n-1)$-form $\hat{\Omega} := N\tilde{\omega}^{n-1} + (-\Omega_E)^{n-1}$ is strictly positive definite for any $N > N_{\max}$. By a result of Michelsohn, in [3, pp. 279–280], which say that any strictly positive definite $(n-1, n-1)$-form $\hat{\Omega}$ can be represented by $\hat{\Omega} = \hat{\omega}^{n-1}$ for some Kähler form $\hat{\omega}$ on $\hat{M}$.

In final, we only needs to show that there exists a closed 1-form such that $\hat{\omega}$ satisfies Eq. (1). Since the supports of $\tilde{\theta}$ and Ω_E are disjoint, so $\tilde{\theta} \wedge \Omega_E = 0$. In addition, $d\hat{\omega}^{n-1} = d(N\tilde{\omega}^{n-1} + (-\Omega_E)^{n-1}) = Nd\tilde{\omega}^{n-1}$, we obtain that

$$\begin{aligned} d\hat{\omega}^{n-1} &= dN\tilde{\omega}^{n-1} = (n-1)\tilde{\theta} \wedge N\tilde{\omega}^{n-1} \\ &= (n-1)\tilde{\theta} \wedge N\tilde{\omega}^{n-1} + (n-1)\tilde{\theta} \wedge (-\Omega_E)^{n-1} \\ &= (n-1)\tilde{\theta} \wedge \hat{\omega}^{n-1}. \end{aligned}$$

This show that there exists a closed 1-form $\hat{\theta} := \tilde{\theta}$ such that $\hat{\omega}$ satisfying

$$d\hat{\omega}^{n-1} = (n-1)\hat{\theta} \wedge \hat{\omega}^{n-1}. \tag{4}$$

This completes the proof of Theorem 1.1.

Acknowledgements

The authors sincerely thanks Bohui Chen, Xiaojun Chen, An-Min Li, Guosong Zhao and Bin Zhang for their encouragement and supports.

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