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An explicit semi-factorial compactification of the Néron model

Une compactification semi-factorielle explicite du modèle de Néron

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ABSTRACT

C. Pépin recently constructed a semi-factorial compactification of the Néron model of an Abelian variety using the flattening technique of Raynaud–Gruson. Here we prove that an explicit semi-factorial compactification is a certain moduli space of sheaves — the family of compactified Jacobians.

Résumé

C. Pépin a construit récemment une compactification semi-factorielle du modèle de Néron d’une variété abélienne en utilisant les techniques de platification de Raynaud–Gruson. Nous montrons ici qu'une compactification semi-factorielle explicite constitue un certain espace de modules de faisceaux — la famille de jacobiens compacifiés.

We prove that the family of compactified Jacobians is a semi-factorial compactification of the Néron model of the Jacobian. Semi-factoriality is a weakening of factoriality, the condition that the local rings are unique factorization domains. In [18], Pépin introduced the condition and proved that the Néron model of an Abelian variety $A_K$ over the field of fractions $K$ of a discrete valuation ring $R$ admits a semi-factorial compactification.

Pépin constructed the compactification using the flattening technique of Raynaud–Gruson [19]. We give an alternative construction when $A_K = J_K$ is a Jacobian satisfying suitable hypotheses. We prove that an explicit semi-factorial compactification is given by a compactification of $J_K$ as a moduli space — by the family of compactified Jacobians.

What is the compactified Jacobian? Suppose $A_K = J_K$ is the Jacobian of the smooth curve $X_K$. The curve $X_K$ extends to a regular model $X/S$ over $S = \text{Spec}(R)$. The Jacobian $J_K$ is the moduli space of degree 0 line bundles on $X_K$, and we can try to extend it to a family $\tilde{J}/S$ by adding over the point $0 \in S$ a moduli space of sheaves on $X_0$. When $X_0$ is geometrically integral, we can extend $J_K$ by adding the moduli space of degree 0 rank 1, torsion-free sheaves on $X_0$, and this extended family is the family of compactified Jacobians.

The line bundle locus $J/S$ in a family of compactified Jacobians $\tilde{J}/S$ is canonically isomorphic to the Néron model of $J_K$ by (a special case of) [12, Theorem 3.9], a result that extends earlier work on the topic [17,3–6,15]. Compactified Jacobians

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are proper by construction, so $\overline{J}/S$ is a compactification of the Néron model. When the Picard rank of $J_K$ is 1, $\overline{J}/S$ has the desirable properties studied by Pépin:

**Main Theorem.** The Altman–D’Souza–Kleiman family of compactified Jacobians $\overline{J}/S$ is a semi-factorial model of the Néron model provided the Picard rank of $J_K$ is 1.

The Main Theorem includes the explicit hypothesis that $J_K$ has Picard rank 1 and the implicit hypothesis that the special fiber $X_0$ is geometrically integral. How are these hypotheses used? When the hypothesis that $X_0$ is geometrically integral fails, the Altman–D’Souza–Kleiman family $\overline{J}/S$ is not defined because the moduli space of degree 0 rank 1, torsion-free sheaves on $X_0$ is badly behaved. A well-behaved space can be recovered by imposing e.g. a stability condition, but the proof we give here does not immediately apply to these more general spaces. In proving the Main Theorem, we use the property that translation $\tau_{\alpha}: J_K \to J_K$ by a point $\alpha \in J_K(K)$ extends to an automorphism $\overline{J} \to \overline{J}$. It is not known if $\overline{J}$ has this extension property when $X_0$ is reducible; the issue is that, when $X_0$ is reducible, the tensor product of two slope semi-stable line bundles can fail to be semi-stable.

The hypothesis that $J_K$ has Picard rank 1 is used to assert that the Néron–Severi group $\text{NS}(J_K)$ is generated by classes that extend to $\overline{J}$. Under the rank 1 hypothesis, $\text{NS}(J_K)$ is generated by the class of the theta divisor, and Esteves and Soucaris have (independently) shown that this divisor extends. In general, when $\text{NS}(J_K)$ is generated by classes that extend, our proof shows that $\overline{J}$ is semi-factorial, and it would be desirable to have more general results describing when classes in $\text{NS}(J_K)$ extend.

1. Preliminaries

Here we collect results from the literature. Fix a discrete valuation ring (or DVR for short) $R$ with field of fractions $K$ and residue field $k(0)$. Set $S = \text{Spec}(R)$ and $0 = \text{Spec}(k(0))$. We fix a smooth curve $X_K/\text{Spec}(K)$ (i.e. a $K$-scheme of pure dimension 1 that is proper, smooth, and geometrically connected over $K$) that we assume has genus $g \geq 1$ and study the associated Jacobian $J_K/\text{Spec}(K)$. The Jacobian is a $g$-dimensional Abelian variety that represents the étale sheaf parameterizing degree 0 line bundles on $X_K$, and it extends to the Néron model $J/K$, a certain (possibly nonproper) $S$-scheme. We omit the definition, but one consequence, which we will use, is that the restriction map $J(S) \to J_K(K)$ is surjective, i.e. the weak Néron Mapping Property holds.

To study compactifications of $J_K$, we make the following definitions.

**Definition 1.** An $S$-scheme $V/S$ is semi-factorial if the restriction map

$$\text{Pic}(V) \to \text{Pic}(V_K)$$

on Picard groups is surjective.

If $V/S$ is separated and of finite type over $S$, then an $S$-compactification of $V/S$ is a proper $S$-scheme $\overline{V}/S$ and an $S$-immersion $V \to \overline{V}$ with dense image. An $S$-compactification is a semi-factorial model if $\overline{V}/S$ is flat and projective over $S$, normal, and semi-factorial. A semi-factorial model is a regular model if $\overline{V}$ is a regular scheme.

Corollaire 6.4 of [18] states that the Néron model $J/S$ admits a semi-factorial model. In fact, the Corollaire states that the semi-factorial model can be chosen to have certain desirable base-change properties, which we discuss in Remark 5.

The curve $X_K$ admits a regular model $X/S$ because resolution of singularities holds for arithmetic surfaces [14]. Lipman’s result is stated for $R$ excellent, but the argument on [7, page 87] shows that this hypothesis can be removed.) For the remainder of this paper, we fix a regular model $X/S$ satisfying

**Assumption.** $X/S$ is a regular model of $X_K$ with geometrically integral special fiber.

With this assumption, the Altman–D’Souza–Kleiman family of compactified Jacobians $\overline{J}/S$ associated to $X/S$ is defined. The family of compactified Jacobians is an $S$-scheme $\overline{J}/S$ that is projective over $S$ and represents the étale sheaf parameterizing families of degree 0 rank 1, torsion-free sheaves on $X/S$ [1, (8.10) Theorem]. (Under more restrictive hypotheses, this is [8, Theorem II.4.1].) The line bundle locus in $\overline{J}$ is an open subscheme $J$ that is the Néron model of $J_K$ [12, Theorem 3.9].

We now recall the definition of the Néron–Severi group and the Picard scheme of $J_K$. The Picard scheme $\text{Pic}(J_K/K)/\text{Spec}(K)$ is a $K$-group scheme that is locally of finite type over $K$ and represents the étale sheaf parameterizing line bundles on $J_K$. The line bundles that are algebraically equivalent to zero are parameterized by the identity component $\text{Pic}^0(J_K/K)$ of the Picard scheme, which is an open and closed $K$-subgroup scheme that is of finite type over $K$.

Algebraic equivalence classes of line bundles on $J_K$ form the Néron–Severi group, which is defined as

$$\text{NS}(J_K) := \frac{\text{Pic}(J_K/K)(K)}{\text{Pic}^0(J_K/K)(K)}$$
for $\bar{K}$ a fixed algebraic closure of $K$. This group is finitely generated, hence has a well-defined rank called the Picard rank.

The Picard rank of $J_K$ is at least 1. Indeed, $J_K$ admits a special type of divisor: the classical theta divisor. If $N_{\bar{K}}$ is a line bundle of degree $g - 1$ on $X_K$, then

$$\Theta_{\bar{K}} := \{ [\mathcal{L}_{\bar{K}}] ; h^0(X_{\bar{K}}, \mathcal{L}_{\bar{K}} \otimes N_{\bar{K}}) \neq 0 \} \subset J_{\bar{K}}$$

is an ample divisor that defines a principal polarization. That is, the homomorphism

$$\phi : J_{\bar{K}} \to \text{Pic}^0(J_{\bar{K}})$$

defined by

$$\phi(a) = \mathcal{O}_{J_{\bar{K}}} (\tau_a^* (\Theta_{\bar{K}}) - \Theta_{\bar{K}})$$

is an isomorphism. Here $\tau_a$ is the translation-by-$a$ map.

The divisor $\Theta_{\bar{K}}$ depends on the choice of $N_{\bar{K}}$, but its image in the Néron–Severi group is independent of the choice, and we denote this common image by $\theta$. Because $\Theta_{\bar{K}}$ is a principal polarization, $\theta$ is nonzero, and furthermore:

**Lemma 2.** The class $\theta$ freely generates $\text{NS}(J_{\bar{K}})$ when the Picard rank of $J_K$ is 1.

**Proof.** If $J_{\bar{K}}$ has Picard rank 1, then the Néron–Severi group $\text{NS}(J_{\bar{K}})$ is cyclic because it is torsion-free [16, Corollary 2, page 178], so we may fix a generator $c$. Writing $\theta = n \cdot c$ for some $n \in \mathbb{Z}$, we have

$$n^g \cdot (c^g/g!) = \theta^g/g! = 1 \text{ by the Riemann–Roch Formula.}$$

So $n^g$ divides 1 and hence $n = \pm 1$. ∎

### 2. Proof of the Main Theorem

Here we prove that $\bar{J}/S$ is a semi-factorial model of the Néron model provided the Picard rank of $J_K$ is 1.

**Lemma 3.** $\bar{J} \to S$ is flat, and $\bar{J}$ is Cohen–Macaulay and normal.

**Proof.** Theorem (9) of [2] states that $\bar{J} \to S$ is flat with Cohen–Macaulay fibers. (That theorem includes the hypothesis that $X$ lies on an $S$-smooth family of surfaces, but we can reduce to this case by arguing as in the proof of [10, Lemma 3.4].) Since $S$ is Cohen–Macaulay, we can conclude that $\bar{J}$ itself is Cohen–Macaulay.

We prove $\bar{J}$ is normal using Serre’s criteria. To verify the criteria, we need to show that Condition R1 holds. The line bundle locus $J_0 \subset \bar{J}$ is dense in the special fiber by [2, Theorem (9)], so the line bundle locus $J \subset \bar{J}$ in the total space contains all codimension 1 points. The locus $J$ is contained in the smooth locus of $\bar{J}/S$, hence in the regular locus of $\bar{J}$, and so Condition R1 is satisfied. ∎

**Proof of the Main Theorem.** By Lemma 3 we just need to show that $\bar{J}/S$ is semi-factorial, i.e.

$$\text{Pic}(\bar{J}) \to \text{Pic}(J_K)$$

is surjective.

First, assume that $X$ admits a line bundle $\mathcal{N}$ with fiber-wise degree $g - 1$. Then the set $[[\mathcal{L}]] \in \bar{J} : h^0(X, \mathcal{L} \otimes \mathcal{N}) \neq 0 \subset \bar{J}$ is the support of a relatively effective divisor $\Theta$ that extends the classical theta divisor by [20, Theorem 13] (or [9, page 184]). In particular, $\mathcal{O}_{J_K}(\Theta_{\bar{K}})$ lies in the image of (3).

That image also contains all line bundles algebraically equivalent to zero. Indeed, the polarization isomorphism $\phi$ from Eq. (2) is defined over $K$, so if $\mathcal{L}_K$ is a line bundle on $J_K$ that is algebraically equivalent to zero, then we can write $[\mathcal{L}_K] = \phi(a_K)$ for some $a_K \in J_K(K)$. Here $[\mathcal{L}_K] \in \text{Pic}^0(J_K/K)(K)$ is the point represented by $\mathcal{L}_K$. The $S$-scheme $J/S$ satisfies the Néron Mapping Property (by e.g. [12, Theorem 3.9]), so $a_K \in J_K(K)$ is the restriction of some $a \in J(S)$. The line bundle locus $J$ acts on $\bar{J}$ (by tensor product), so translation $\tau_a : \bar{J} \to \bar{J}$ by $a$ is well-defined, and the line bundle $\mathcal{L} := \mathcal{O}_{\bar{J}}(\tau_a^* (\Theta) - \Theta)$ extends $\mathcal{L}_K$.

We have now shown that the image of (3) contains both $\mathcal{O}_{J_K}(\Theta_{\bar{K}})$ and the line bundles algebraically equivalent to zero. Together these line bundles generate $\text{Pic}(J_K)$ by Lemma 2, so (3) is surjective, proving the theorem in the special case that an $\mathcal{N}$ exists.

In the general case, we argue as follows. Given a line bundle $\mathcal{L}_K$ on $J_K$, we can extend $\mathcal{L}_K$ to a family $\mathcal{L}$ of rank 1, torsion-free sheaves on $\bar{J}$ (by e.g. the S-projectivity of the relevant compactified Picard scheme). There exists a line bundle $\mathcal{N}$ with fiber-wise degree $g - 1$ on $X_T$ for some étale cover $T \to S$ with $T$ the spectrum of a dvr because $X_0$ is geometrically reduced. Say $L$ is the field of fractions of the dvr $\Gamma(T, \mathcal{O}_T)$. The base-change $X_T$ remains regular, so $\mathcal{L}_L$ extends to a line bundle on $\bar{J}_T$. This extension must equal $\mathcal{L}_T$ (by e.g. the S-separateness of the relevant compactified Picard scheme), so $\mathcal{L}_T$ and hence $\mathcal{L}$ must be a line bundle. ∎
Remark 4. Does \( \mathcal{J} \) satisfy stronger conditions than semi-factoriality? Typically \( \mathcal{J} \) does not satisfy the condition of regularity. Let \( K = \mathbf{Q}, R = \mathbf{Z} \cup \{0\} \) (the localization of \( \mathbf{Z} \) at \( 3 \)). \( S = \text{Spec}(R) \), and \( X/S \) the minimal proper regular model of the affine curve \( \text{Spec}(R[x,y]/(y^2 - x^2(x-1)(x^2+1)-3)) \). The family \( X/S \) is a family of genus \( 2 \) curves with special fiber \( X_0 \) a rational curve with two nodes. Consider the family of compactified Jacobians \( \mathcal{J}/S \) associated to \( X/S \).

If \( v : \mathbf{P}^1 \to X_0 \) is the normalization, then \( \mathcal{J} \) has a singularity at the rank \( 1 \), torsion-free sheaf \( \mathcal{I} := v_* \mathcal{O}(-2) \). The singularity of \( \mathcal{J} \) at \( \mathcal{I} \) is computed in [13]. The sheaf \( \mathcal{I} \) fails to be locally free at 2 nodes, so by [13, Lemma 6.2] the completed local ring is isomorphic to

\[
\tilde{\mathcal{O}} = \tilde{\mathcal{R}}[[a_1, b_1, a_2, b_2]]/(a_1 b_1 - 3, a_2 b_2 - 3).
\]

This ring not only fails to be regular, but it also fails to be factorial. (The height 1 prime \( (a_1, a_2) \) is nonprincipal because the images of \( a_1, a_2 \) in the quotient \( (3, a_1, b_1)/(3, a_2, b_2) \) are linearly independent.)

However, \( \mathcal{J}/S \) is semi-factorial. Indeed, by the Main Theorem, we just need to show that \( J(K) = J_0 \) has Picard rank \( 1 \), and we do so as follows. The Néron–Severi group \( \text{NS}(J_0) \) injects into the endomorphism ring \( \text{End}(J_0) \), and we compute this endomorphism ring by relating it to the endomorphism ring of the reduction of \( J_0 \) at a prime of good reduction.

Both the curve \( X_0 \) and its Jacobian \( J_0 \) have good reduction at the primes \( p = 5, 13 \), as can be seen by reducing the equation \( y^2 = x^2(x-1)^2(x^2+1)+3 \) mod \( p \). Using this equation to naively count \( F_p \)-points, we compute that the characteristic polynomial \( f_p \) of the Frobenius endomorphism of \( J_{F_p} \) is

\[
\begin{align*}
    f_5 &= x^4 - 2x^3 + 3x^2 - 10x + 25, \\
    f_{13} &= x^4 + 7x^3 + 35x^2 + 91x + 169.
\end{align*}
\]

Applying [11, Theorem 6] to these polynomials, we get that the reduction \( J_{F_p} \) is absolutely simple for \( p = 5, 13 \), so \( \mathbf{Q} \otimes \text{End}(J_{F_p}) = \mathbf{Q}[x]/(f_p) \).

The reduction map injects \( \mathbf{Q} \otimes \text{End}(J_0) \) into \( \mathbf{Q} \otimes \text{End}(J_{F_p}) \) for \( p = 5, 13 \). A computation shows that the discriminant of \( \mathbf{Q}[x]/(f_5) \) is coprime to the discriminant of \( \mathbf{Q}[x]/(f_{13}) \), and \( \mathbf{Q} \) has no nontrivial unramified extensions, so the only field contained in both \( \mathbf{Q}[x]/(f_5) \) and \( \mathbf{Q}[x]/(f_{13}) \) is \( \mathbf{Q} \). In particular, \( \text{End}(J_{F_p}) = \mathbf{Z} \). This example was suggested to the author by Bjorn Poonen.

Remark 5. Corollaire 6.4 of [18] proves that a semi-factorial model \( \mathcal{J}/S \) of \( J_K \) can be chosen to be well-behaved with respect to certain dvr extensions. To be precise, given morphisms \( T_1 \to S, \ldots, T_n \to S \) corresponding to extensions of \( R \) contained in the strict henselization \( R^{\mathrm{sh}} \), a semi-factorial model \( \mathcal{J}/S \) can be chosen so that \( \mathcal{J}/T_i \) is a semi-factorial model when \( T \to S \) equals some \( T_i \to S \) or a morphism corresponding to a “permise” dvr extension.

The family \( \mathcal{J}/S \) of compactified Jacobians satisfies this condition. In fact, it satisfies a stronger condition. By definition the family of the model of compactified Jacobians commutes with arbitrary base change, so if \( T \to S \) is a morphism that corresponds to a dvr extension, then \( \mathcal{J}_T/T \) is a semi-factorial model of the Néron model provided \( X_T \) is regular. The scheme \( X_T \) is regular when \( T \to S \) is one of the morphisms considered by Pépin or more generally when \( T \to S \) is regular and surjective (see [18, Remarque 5.5]).

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