Differential geometry

Harmonic vector fields on Landsberg manifolds

Champs de vecteurs harmoniques sur les variétés landsbergiennes

Alireza Shahi 1, Behroz Bidabad *

Faculty of Mathematics, Amirkabir University of Technology (Tehran Polytechnic), 424 Hafez Avenue, 15914 Tehran, Iran

A R T I C L E    I N F O

Article history:
Received 12 February 2014
Accepted after revision 4 August 2014
Available online 27 August 2014
Presented by the Editorial Board

A B S T R A C T

Let $(M, F)$ be a compact boundaryless Landsberg manifold. In this work, a necessary and sufficient condition for a vector field on $(M, F)$ to be harmonic is obtained. Next, on a compact boundaryless Finsler manifold of zero flag curvature, a necessary and sufficient condition for a vector field to be harmonic is found. Furthermore, the nonexistence of harmonic vector fields on a compact Landsberg manifold is studied and an upper bound for the first de Rham cohomology group is obtained.

© 2014 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

R É S U M É

Soit $(M, F)$ une variété landsbergienne compacte sans bord. Dans cet article, il est obtenu une condition nécessaire et suffisante pour qu’un champ de vecteurs sur $(M, F)$ soit harmonique. On donne ensuite un énoncé analogue sur une variété finslérienne compacte sans bord. En outre, on étudie la non-existence de champs de vecteurs harmoniques sur les variétés landsbergiennes compactes et, enfin, une borne supérieure pour le premier groupe de cohomologie de de Rham est obtenue.

© 2014 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

On a 2-dimensional Riemannian manifold, a harmonic vector field is a vector field for which divergence and curl operators vanish. In a compact Riemannian manifold, the existence of harmonic vector fields is closely related to the sign of the Ricci curvature and the topology of the underlying manifold. Bochner in [6] used the Laplace–Beltrami operator to prove some theorems on nonexistence of harmonic vector fields on compact Riemannian manifolds with positive or negative Ricci curvature. For instance, he proved that on a compact Riemannian manifold without boundary and with positive Ricci curvature, there is no vector field for which divergence and differential operators vanish simultaneously. Yano in [8] obtained a general formula and proved some of Bochner’s theorems in compact Riemannian manifolds. Next, Wu in [7] proved that in Bochner’s theorems, the assumption of positivity or negativity of the Ricci curvature can be replaced by quasi-positivity or

In the present work we generalize some results of Yano for Landsberg manifolds. More intuitively a necessary and sufficient condition for a vector field to be harmonic is obtained, and the nonexistence of harmonic vector fields under certain conditions on a Landsberg manifold is proved. Moreover, on an $n$-dimensional compact Landsberg manifold, the dimension of the first de Rham cohomology group is at most $2n - 1$ under a certain condition.

2. Preliminaries

Let $(M, F)$ be a Finsler manifold, $\pi : TM_0 \to M$ the bundle of non-zero tangent vectors and $\pi^*TM$ the pullback bundle. We adopt here often the notations and the terminology of [2] and sometimes those of [4].

Denote the covariant derivatives of Cartan and Berwald connections by $\nabla$ and $\partial$ respectively. It is well known that the Whitney sum $TTM_0 = HTM \oplus VTM$, where $HTM$ and $VTM$ are horizontal and vertical bundles, respectively, and for any $X \in TTM_0$ we have $X = HX + VX$. Let $X$ and $Y$ be two sections on $\pi^*TM$. The relations between Cartan and Berwald connections are given by

\[ D_{HX}Y = \nabla_{HX}Y + y^i(\nabla_i T)(X, Y), \]
\[ D_{VX}Y = \partial VX, \]

where $T$ is the Cartan tensor with the components $T_{kij} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}$ and $y = \frac{\partial y^i}{\partial x^k} \in TXM$. By means of (1) we have $D_k g_{ij} = -2V_0 T_{kij}$, where index 0 denotes the contracted multiplication by $y$, hence $D_0 g_{ij} = \theta^k D_k g_{ij} = -2\theta^k V_0 T_{kij} = 0$. Eq. (2) is written locally $D_j y^i = \frac{\partial y^i}{\partial y^j}$. The Ricci identities for Berwald connection are

\[ D_i D_k X^l - D_k D_i X^l = X^j R_{j i k}^l - \frac{\partial X^l}{\partial y^j} R_{0 i k}^j, \]
\[ D_i D_k X_j - D_k D_i X_j = -X_l R_{j i k}^l - \frac{\partial X_j}{\partial y^l} R_{0 i k}^l. \]

The Finsler manifold $(M, F)$ is called Landsberg manifold if the $h\nu$-curvature $V$ vanishes everywhere or equivalently $V_0 T = 0$. Let $X = X^i(x) \frac{\partial}{\partial x^i}$ be a vector field on $M$. One can associate with $X$ the 1-form $\tilde{X}$ on $SM$ defined by $\tilde{X} = X(z) dx^i + X_i dz^i$, where $X_i = \frac{\partial X_i}{\partial y^j}$. The horizontal part of its associated 1-form on $SM$ is denoted again in this paper by $X = X(z) dx^i$, where $z \in SM$, cf. [2], p. 231, and the restriction of the first de Rham cohomology group to these 1-forms by $H^1_{d\bar{d}}(SM)$. The differential and co-differential operators of the horizontal 1-form $X$ are given by

\[ dX = \frac{1}{2}(D_i X_j - D_j X_i) dx^i \wedge dx^j - \frac{\partial X_i}{\partial y^j} dx^i \wedge dy^j, \]
\[ \delta X = -(\nabla X_j - X_j V_0 T^j) = -g^{ij} D_i X_j, \]

where the co-differential operator $\delta$ is the formal adjoint of $d$, in the global scalar product over $SM$, cf. [2] pp. 223 and 239. Let $(M, F)$ be a compact Finsler manifold without boundary, the divergence formula for a horizontal 1-form $X = X(z) dx^i$ is given by

\[ \int_S (\delta X) \eta = -\int_S (g^{ij} D_i X_j) \eta = 0, \]

where $\eta$ is a volume form on $SM$ defined by

\[ \eta(g) = (-1)^{\frac{(n-1)}{2}} \frac{(n-1)!}{n!} \omega \wedge d\omega \wedge \ldots \wedge d\omega, \]

$\omega = u_i dx^i$ is the 1-form corresponding to a unitary vector field $u : M \to SM$.

3. Harmonic vector fields

Let $(M, F)$ be a Finsler manifold, the vector field $X$ on $M$ is said to be $\text{harmonic}$ if its corresponding horizontal 1-form on $SM$ satisfies $\Delta X = d\bar{d}(X) + d\delta(X) = 0$ or $dX = 0$ and $\delta X = 0$, equivalently:

\[ D_i X_j = D_j X_i, \]
\[ g^{ij} D_i X_j = 0, \]
\[ \frac{\partial X_i}{\partial y^j} = 0. \]
**Theorem 3.1.** If the vector field $X$ on a compact Landsberg manifold without boundary satisfies $g^{jk} D_k D_j X^i = (R^i_k + T^i_r R^r_{0k}) X^k$ and $\frac{\partial X_i}{\partial y^j} = 0$, then $X$ is harmonic. The converse is true if $T^i_r R^r_{0k} X^k = 0$.

**Proof.** Let $g^{jk} D_k D_j X^i = (R^i_k + T^i_r R^r_{0k}) X^k$ and $\frac{\partial X_i}{\partial y^j} = 0$. By covariant derivative, we have:

$$D_j (X^k D_k X^i) = X^k D_j D_k X^i + D_k X^i D_j X^k.$$  \hspace{1cm} (9)

Note that

$$\frac{\partial X^i}{\partial y^j} = D^i_j X^i = D^i_j \left( X_h g^{ih} \right) = X_h D^i_j g^{ih} + g^{ih} D^i_j X_h = X_h T^i_j + g^{ih} \frac{\partial X_h}{\partial y^j}. \hspace{1cm} (10)$$

By means of the torsion freeness of the Berwald connection, the assumption $\frac{\partial X_i}{\partial y^j} = 0$, (3) and (10) we obtain:

$$D_1 D_k X^i - D_k D_1 X^i = R^i_{jk} X^j - X_h T^i_j R^r_{0k}.$$  \hspace{1cm} (11)

By contraction of the above equation with respect to $i$ and $l$, we obtain:

$$D_j D_k X^i - D_k D_j X^i = R_{jk} X^j + X^j T^i_{ij} R^r_{0k},$$  \hspace{1cm} (12)

or equivalently

$$D_j D_k X^i = D_k D_j X^i + (R_{jk} + T^i_{ij} R^r_{0k}) X^i.$$  \hspace{1cm} (13)

Replacing (11) in (9), we find:

$$D_j (X^k D_k X^i) = X^k D_k D_j X^i + (R_{jk} + T^i_{ij} R^r_{0k}) X^j X^k + D_k X^i D_j X^k.$$  \hspace{1cm} (12)

The covariant derivative of $X^k D_j X^i$ yields:

$$D_k (X^k D_j X^i) = X^k D_k D_j X^i + D_k X^i D_j X^k.$$  \hspace{1cm} (13)

Using (12) and (13) we have:

$$D_j (X^k D_k X^i) - D_k (X^k D_j X^i) = (R_{jk} + T^i_{ij} R^r_{0k}) X^j X^k + D_k X^i D_j X^k - D_k X^k D_j X^i.$$  \hspace{1cm} (14)

Since $M$ is compact and the two terms in the left-hand side of (14) are divergence, by integration of (14) over $SM$ and using divergence formula (7), we obtain:

$$\int_{SM} \left[ (R_{jk} + T^i_{ij} R^r_{0k}) X^j X^k + D_k X^i D_j X^k - D_k X^k D_j X^i \right] \eta = 0.$$  \hspace{1cm} (15)

Let us consider the function $\phi = X_i X^i$ on $SM$, we have:

$$g^{jk} D_k D_j \phi = g^{jk} [D_k X_i D_j X^i + X_i D_k D_j X^i + D_k X^i D_j X^i + X^i D_k D_j X^i].$$  \hspace{1cm} (16)

For a Landsberg manifold $D_k g_{ij} = 0$, hence $X^i D_k D_j X_i = X_i D_k D_j X^i$, therefore (16) reduces to

$$g^{jk} D_k D_j \phi = 2 X_i g^{jk} D_k D_j X^i + 2 D^i D_j X^i.$$  \hspace{1cm} (17)

The left-hand side in (17) is divergence, hence by integration over $SM$ and using divergence formula (7), we obtain:

$$\int_{SM} \left[ X_i g^{jk} D_k D_j X^i + D^i D_j X^i \right] \eta = 0.$$  \hspace{1cm} (18)

By subtracting (15) and (18) we get:

$$\int_{SM} \left[ (X_i g^{jk} D_k D_j X^i - (R_{jk} + T^i_{ij} R^r_{0k}) X^i X^k) + (D^i D_j X^i X^k - D_k X^i D_j X^k) + D_k X^k D_j X^i \right] \eta = 0.$$  \hspace{1cm} (19)

Note that $(R_{jk} + T^i_{ij} R^r_{0k}) X^i X^k = (R^i_k + T^i_r R^r_{0k}) X^i X^k$, hence relation (19) is equivalent to
\[
\int_{SM} \left[ X_i (g^{jk} D_k D_j X^i - (R^i_k + \gamma_{ij} R^i_{0_k}) X^k) + \frac{1}{2} (D_j X^k - D^k X^j)(D_j X_k - D_k X_j) + D_k X^i D_j X^j \right] \eta = 0. \tag{20}
\]

By assumption, the first term in (20) is zero. The next two terms \( \frac{1}{2} \| (D_j X_k - D_k X_j) \|^2 \) and \( (D_j X^j)^2 \) are positive or zero, hence by (20) are zero. Therefore we obtain \( D_j X_k - D_k X_j = 0 \) and \( D_j X^j = 0 \), thus by definition \( X \) is a harmonic vector field on \( M \). Conversely, let \( X \) be a harmonic vector field on a Landsberg manifold and \( T_i^u R^u_{0_k} X^k = 0 \). Replacing \( D_i X_j = D_i X_i \) and \( \partial X_i / \partial y^j = 0 \) in (4) yields

\[
D_i D_k X_j - D_k D_j X_i = -R^i_{jk} X_i. \tag{21}
\]

Multiplying (21) by \( g^{jk} \) on Landsberg manifold, we have:

\[
D_i D^j X_j - g^{jk} D_k D_j X_i = -R^i_{jk} X_i, \quad \text{or} \quad g^{jk} D_k D_j X^i = R^i_{jk} X^i. \]

Hence proof is complete. \( \square \)

**Example 1.** Let \( (M, F) \) be a simply connected, compact Landsberg manifold with positive constant flag curvature. It is well known that \( (M, F) \) reduces to a Riemannian manifold and \( T_i^u = 0 \), cf. [5] or [2] p. 129. Therefore a vector field \( X \) on \( M \) is harmonic if and only if \( g^{jk} D_k D_j X^i = R^i_{jk} X^i \) and \( \partial X_i / \partial y^j = 0 \).

**Corollary 3.2.** Let \( (M, F) \) be a compact Finsler manifold without boundary and with zero flag curvature. A necessary and sufficient condition for a vector field \( X \) on \( M \) to be harmonic is \( g^{jk} D_k D_j X^i = 0 \) and \( \partial X_i / \partial y^j = 0 \).

**Proof.** It is well known that any compact Finsler manifold without boundary of zero flag curvature is a Landsberg manifold and \( hh \)-curvature of Berwald connection vanishes, cf. [2], p. 164 and [4], p. 328. Hence the proof is a direct conclusion of the above theorem. \( \square \)

**Theorem 3.3.** Let \( (M, F) \) be a compact Landsberg manifold without boundary and \( X \) a harmonic vector field on \( M \). If \( (R_{jk} + T_{ij} R^i_{0_k}) X^j X^k \geq 0 \), then the covariant derivatives of \( X \) with respect to the Cartan and Berwald connections vanish. Moreover, there exists no non-zero harmonic vector field on \( (M, F) \) which satisfies the relation \( (R_{jk} + T_{ij} R^i_{0_k}) X^j X^k > 0 \).

**Proof.** Let \( X \) be a harmonic vector field on \( M \). By definition,

\[
g^{ij} D_i X_j = 0, \quad D_i X_j = D_j X_i, \quad \partial X_i / \partial y^j = 0. \tag{22}
\]

Therefore on a Landsberg manifold:

\[
D_k X^k = g^{kj} D_k X_j = 0. \tag{23}
\]

On the other hand by means of (22) we have:

\[
D_k X^j D_j X^k = g^{ij} D_k X_j D_j X^k = g^{ij} D_i X_k D_j X^k = \| D_i X_k \|^2 \geq 0. \tag{24}
\]

Substituting (23) and (24) in (15) we obtain:

\[
\int_{SM} \left( (R_{jk} + T_{ij} R^i_{0_k}) X^j X^k + \| D_i X_k \|^2 \right) \eta = 0. \tag{25}
\]

If the vector field \( X \) satisfies in \( (R_{jk} + T_{ij} R^i_{0_k}) X^j X^k \geq 0 \), then by (25) we find \( \| D_i X_k \|^2 = 0 \) and hence \( \forall \ i X_k = D_j X_k = 0 \). Moreover, by (25) there exist no non-zero harmonic vector field on \( M \) which satisfies \( (R_{jk} + T_{ij} R^i_{0_k}) X^j X^k > 0 \). This completes the proof. \( \square \)

**Theorem 3.4.** Let \( (M, F) \) be an \( n \)-dimensional compact Landsberg manifold without boundary and \( X \) a harmonic vector field on \( M \). If \( (R_{jk} + T_{ij} R^i_{0_k}) X^j X^k \geq 0 \), then the first de Rham cohomology group of \( SM \) satisfies \( \dim H^1_{dR} (SM) \leq 2n - 1 \). Moreover, if \( (R_{jk} + T_{ij} R^i_{0_k}) X^j X^k > 0 \), then \( H^1_{dR} (SM) = 0 \).

**Proof.** Let \( X \) be a harmonic vector field on \( (M, F) \) endowed with a Berwald connection which satisfies \( (R_{jk} + T_{ij} R^i_{0_k}) X^j X^k \geq 0 \). By definition of harmonic vector fields and Theorem 3.3, we have \( D_i X_j = D_j X_i = 0 \). Therefore the
1-form \( X = X_i(z)dx^i \) is parallel with respect to the Berwald connection on \( SM \). It is well known that a parallel 1-form on a manifold is determined by its value at a point on the underlying manifold. Hence the dimension of the vector space of parallel 1-forms is at most equal to the dimension of the cotangent \( T^*_x(SM) \), that is \( 2n - 1 \). Therefore \( \dim H^1_{dR}(SM) \leq 2n - 1 \). Next, if \( (R_{jk} + T_{ij} R_{0 k}^i)X^j X^k > 0 \), then by means of Theorem 3.3 there is no non-zero harmonic vector field \( X \) which satisfies the above strict inequality. Hence \( SM \) has no nontrivial harmonic 1-form. Thus \( H^1_{dR}(SM) = 0 \) and the proof is complete. \( \square \)

References