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Partial differential equations

The method of differential contractions

La méthode des contractions différentielles

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ABSTRACT

In this Note, we present a general and fairly simple method to design families of contractions for nonlinear partial differential equations, either of evolution type, or of stationary type. As a particular example, we apply this method to the porous medium equation, for which we get new contractions. This method opens new directions to explore.

RÉSUMÉ

Dans cette Note, nous présentons une méthode simple et générale pour fabriquer des familles de contractions pour des équations aux dérivées partielles non linéaires, d'évolution, ou bien stationnaires. À titre d'exemple, cette méthode est appliquée à l'équation des milieux poreux, pour laquelle nous obtenons de nouvelles contractions. Cette méthode ouvre de nouvelles voies de recherche à explorer.

1. Introduction

In this Note, we introduce a new tool that we call the method of *differential contractions*. This method allows us to design families of contractions for general PDEs of evolution type or of stationary type. To explain clearly the method, we will focus on a very well-studied case: the porous medium-type equation for $m > 0$, that we normalize (for convenience) as follows:

$$\partial_t h = \Delta\left(\frac{h^m}{m}\right) \quad \text{on} \quad Q = (0, +\infty) \times \Omega \tag{1}$$

where Ω is an open set in dimension $d \geq 1$. We will also consider the stationary analogue:

$$h - \Delta\left(\frac{h^m}{m}\right) = f \quad \text{on} \quad \Omega. \tag{2}$$

The reader will understand the generality of the method that can be applied to a large variety of equations (with possible coefficients depending on space or time coordinates). These equations include the p-Laplacian, the doubly nonlinear

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equation, quasilinear equations like for instance the minimal surface equation, some parabolic systems, and even certain particular hyperbolic systems. The application to some of these equations is contained in [3] and will be presented in a subsequent work [5].

We give the typical contraction results that we can get, but the most interesting is the method itself which is presented in Section 2, and naturally provides new directions to explore. Given two functions $g_i(x)$ for $i=0,1$, we define the distance

$$d_{\alpha,p}(g_1,g_0)=\left(\int_\Omega \left|g_1^\alpha-g_0^\alpha\right|^p \mathrm{d}x\right)^{1/p}$$

for $(\alpha,p)\in K_{|n|}$ and $n=m-1\in(-1,1)$, with the following definition of the set for $n\neq 0$:

$$K_{|n|}=\left\{(\alpha,p)\in(0,+\infty)\times\left[1,1/n^2\right],\ \alpha\in\left[\alpha_-(p),\alpha_+(p)\right],\ \text{with}\ \alpha_\pm(p)=1+\frac{(p-1)}{2p}\left(-1\pm\sqrt{1-n^2p}\right)\right\}.$$

It is possible to see that this set is convex and that the minimal value of α corresponds to the point $(\alpha,p)=(|n|,\frac{2-|n|}{|n|})$, which is related to the classical pressure term h^{m-1} when $m>1$. The maximal value of α corresponds to the point $(\alpha,p)=(1,1)$. For $n=0$, we set $K_0=\{(\alpha,p)\in(0,1]\times[1,+\infty),\ \ \alpha\geq 1/p\}$. For convenience, we present our rigorous results when the open set Ω is a torus, but this particular choice of Ω is absolutely not fundamental.

Theorem 1.1 *(Contraction family for porous medium type equations). Assume that we work on the torus $\Omega=\mathbb{T}^d$ with $d\geq 1$ and that $m-1=n\in(-1,1)$. Let $0\leq h_i^0\in L^\infty(\Omega)$ be two initial data for $i=0,1$. Let us call $h_i\in C([0,+\infty);L^1(\Omega))\cap L^\infty(Q)$ the unique solutions to (3) with initial data h_i^0 for $i=0,1$. Then we have the following contraction in time with $h_i(t)=h_i(t,\cdot)$*

the map $t\mapsto d_{\alpha,p}\big(h_1(t),h_0(t)\big)$ is nonincreasing

if $(\alpha,p)\in K_{|n|}$.

Up to our knowledge, in any dimensions, only contractions in L^1, H^{-1} and the 2-Wasserstein distance are known for solutions to (1) (see [10,8]). Our result provides a new contraction family that can be seen as a generalization of the L^1 contraction. A direct approach to this result will be presented in [4] in the case $\Omega=\mathbb{R}^d$. Note that even for the standard heat equation, our result seems new.

Theorem 1.2 *(Contraction family for the stationary equation). Assume that we work on the torus $\Omega=\mathbb{T}^d$ with $d\geq 1$ and that $m-1=n\in(-1,1)$. Let $0\leq f_i\in L^\infty(\Omega)$ be two data for $i=0,1$. Let $0\leq h_i\in L^\infty(\Omega)$ be the unique solutions to (2) with right-hand side $f=f_i$ for $i=0,1$. Then we have*

$$d_{\alpha,p}(h_1,h_0)\leq d_{\alpha,p}(f_1,f_0)$$

if $(\alpha,p)\in K_{|n|}$.

The proofs of Theorems 1.1 and 1.2 are given in Section 3.

2. The method

Here we present heuristically the method, which is quite elementary.

2.1. The evolution case

At least for smooth positive solutions, we prefer to write Eq. (1) as follows with $n=m-1$:

$$\partial_t h=\operatorname{div}\left(h^n\nabla h\right). \tag{3}$$

2.1.1. Motivation

For two positive functions h_ε and h_0 with the same mass $\int_\Omega h_\varepsilon\,\mathrm{d}x=\int_\Omega h_0\,\mathrm{d}x$, it is well known (see for instance [2,6,7] and the references therein) that the relative entropy $S(h_\varepsilon|h_0)=\int_\Omega \psi(\frac{h_\varepsilon}{h_0})h_0\,\mathrm{d}x$ with $\psi(g)=g\ln g$ plays a key role in the study of the long-time behavior of diffusion equations. Indeed, for $h_\varepsilon=h_0+\varepsilon h_0'+o(\varepsilon)$, where h_0' is a function, we have:

$$S(h_\varepsilon|h_0)\sim\varepsilon^2\int_\Omega \frac{|h_0'|^2}{h_0}\,\mathrm{d}x. \tag{4}$$

Remark 1 *(Relation with the 2-Wasserstein distance).* Note that a simple computation allows us to see that the 2-Wasserstein distance W_2 satisfies in dimension $d=1$: $W_2^2(h_\varepsilon,h_0)\sim\varepsilon^2\int_{\mathbb{R}}\frac{|H_0'|^2}{h_0}\,\mathrm{d}x$, with $H_0'(x)=\int_{-\infty}^x h_0'(y)\,\mathrm{d}y$, which shares some similarities with (4).

2.1.2. Checking the differential contraction

Given now two positive smooth solutions $h_i = h_i(t) = h_i(t,\cdot)$ for $i = 0, 1$, it is interesting to consider a smooth curve h_s of positive smooth solutions to (3) connecting h_0 with h_1 that we parameterize by $s \in [0,1]$. We write for short:

$$h = h_s \quad \text{and} \quad h' = \frac{\mathrm{d}}{\mathrm{d}s} h_s$$

where h' solves the linearized equation:

$$\partial_t h' = \operatorname{div}\big(n h^{n-1} h' \nabla h + h^n \nabla h'\big). \tag{5}$$

We then consider the general differential action

$$S\big(h,h'\big) = \int_\Omega \mathcal{L}\big(h,h'\big)\,\mathrm{d}x$$

that generalizes (4) and is devoted to be non-increasing in time, for a certain Lagrangian $\mathcal{L}$ to determine. To check the differential contraction, we simply compute (dropping the dx in the integral):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} S\big(h,h'\big) &= \int_\Omega \mathcal{L}'_h \partial_t h + \mathcal{L}'_{h'} \partial_t h' \\ &= -\int_\Omega h^n \begin{pmatrix} \nabla \ln h \\ \nabla \ln |h'| \end{pmatrix}^{\mathrm{T}} Q \begin{pmatrix} \nabla \ln h \\ \nabla \ln |h'| \end{pmatrix}\end{aligned} \tag{6}$$

where we have used (3), (5) and integration by parts to get the matrix:

$$Q = \begin{pmatrix} A & C \\ C & B \end{pmatrix} \quad \text{with} \quad \begin{cases} A = h^2 \mathcal{L}''_{hh} + n h h' \mathcal{L}''_{hh'} \\ B = h'^2 \mathcal{L}''_{h'h'} \\ C = h h' \mathcal{L}''_{hh'} + \frac{n}{2} h'^2 \mathcal{L}''_{h'h'}. \end{cases}$$

The goal is then to choose carefully the function $\mathcal{L}$ such that the symmetric matrix Q is nonnegative. Several choices are possible, which lead to more or less exotic contractions. At least, if we think to the homogeneity of our equation, it seems reasonable to try a homogeneous Lagrangian $\mathcal{L}$ as follows (which again generalizes (4)):

$$\mathcal{L}\big(h,h'\big) = h^\beta \frac{|h'|^p}{p}.$$

This gives

$$Q = h^\beta \big|h'\big|^p \bar{Q} \quad \text{with } \bar{Q} = \begin{pmatrix} \frac{\beta(\beta-1)}{p} + n\beta & \beta + \frac{n}{2}(p-1) \\ \beta + \frac{n}{2}(p-1) & p-1 \end{pmatrix}$$

and $-\det \bar{Q} = p^{-1}\{\beta^2 + (p-1)\beta + \frac{n^2 p(p-1)^2}{4}\}$. For $|n| \le 1$, we deduce that this matrix is nonnegative if

$$p \in \big[1, 1/n^2\big] \quad \text{and} \quad \beta \in [\beta_-, \beta_+] \quad \text{with } \beta_\pm = \frac{(p-1)}{2}\big(-1 \pm \sqrt{1 - n^2 p}\big). \tag{7}$$

This shows the fundamental differential contraction

$$\frac{\mathrm{d}}{\mathrm{d}t} S \le 0. \tag{8}$$

2.1.3. Definition of a pseudo-distance

Now, given two positive smooth functions $g_i(x)$ for $i = 0, 1$, we define the set $\Gamma_{g_0}^{g_1}$ of smooth curves $\gamma = (\gamma_s)_{s\in[0,1]}$ such that $\gamma_0 = g_0$, $\gamma_1 = g_1$. We define the pseudo-distance

$$d(g_1, g_0) = \inf_{\gamma \in \Gamma_{g_0}^{g_1}} \mathcal{A}(\gamma) \quad \text{with } \mathcal{A}(\gamma) = c \int_0^1 S\big(\gamma_s, \gamma'_s\big)\,\mathrm{d}s \tag{9}$$

where $c > 0$ is a normalization constant. We recall that $()' = \frac{\mathrm{d}}{\mathrm{d}s}()$, and set $\alpha = 1 + \frac{\beta}{p}$. Therefore, using the fact that $(\gamma_s^\alpha)' = \alpha \gamma_s^{\alpha-1} \gamma'_s$, we get:

$$\mathcal{A}(\gamma) = \int_\Omega \mathrm{d}x \left(\int_0^1 \big|(\gamma_s^\alpha)'\big|^p \,\mathrm{d}s \right) \quad \text{if } c = p\alpha^p$$

with $\alpha > 0$ with our choices of β. We recall that $\gamma_s^\alpha = g_s^\alpha$ for $s = 0, 1$. Then a classical optimization of the convex functional $s \mapsto \int_0^1 \mathrm{d}s |(\gamma_s^\alpha)'|^p$ shows that the infimum in (9) is reached for the straight line $\gamma_s^\alpha = g_0^\alpha + s(g_1^\alpha - g_0^\alpha)$ and then

$$d(g_1, g_0) = \int_\Omega |g_1^\alpha - g_0^\alpha|^p \,\mathrm{d}x = \big(d_{\alpha,p}(g_1, g_0)\big)^p.$$

2.1.4. Conclusion

From (8), we deduce with $h(t) = (h_s(t, \cdot))_{s\in[0,1]}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}\big(h(t)\big) = c\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\big(S\big(h_s(t), h_s'(t)\big)\big)\,\mathrm{d}s \le 0.$$

Therefore, for any $0 \le t_1 < t_2$, we have:

$$d\big(h_1(t_2), h_0(t_2)\big) \le \mathcal{A}\big(h(t_2)\big) \le \mathcal{A}\big(h(t_1)\big).$$

If, finally, we choose at time $t = t_1$ the data $h(t_1)$ such that $\mathcal{A}(h(t_1)) = d(h_1(t_1), h_0(t_0))$, we deduce that

$$\text{the map } t \mapsto d\big(h_1(t), h_0(t)\big) \text{ is non-increasing,}$$

which establishes the expected contraction.

Remark 2 *(Evaluation/optimization of the dissipation term).* The dissipation term $\frac{\mathrm{d}}{\mathrm{d}t}(d(h_1, h_0))$ can be either computed directly as in [4,3], or estimated using an integration $\int_0^1 \mathrm{d}s$ of the right-hand side of (6) and doing some (at least partial) optimization. It would be also interesting to find new associated functional inequalities as in the entropy–entropy dissipation method (see [2,6,7]).

2.2. Adaptation to the stationary case

Similarly to the evolution case, at least for smooth positive solutions, we prefer to write Eq. (2) as follows:

$$h - \mathrm{div}\big(h^n \nabla h\big) = f \quad \text{on} \quad \Omega. \tag{10}$$

We consider two solutions $h_i(x)$ of (10) associated with data f_i for $i = 0, 1$. Then we introduce a curve of functions f_s that coincides with the data f_i for $s = i = 0, 1$, and call h_s the corresponding solutions to (10) with data f_s. We compute:

$$S\big(h, h'\big) - S\big(f, f'\big) = \int_\Omega \mathcal{L}\big(h, h'\big) - \mathcal{L}\big(f, f'\big) \le \int_\Omega (h - f)\mathcal{L}_h'\big(h, h'\big) + \big(h' - f'\big)\mathcal{L}_{h'}'\big(h, h'\big)$$

where we have used the convexity of $\mathcal{L}$ in (h, h') to get the inequality. Indeed, computing the Hessian of $\mathcal{L}$, it is easy to check that the convexity of $\mathcal{L}$ holds for our choices of β in (7). We then conclude using the equations satisfied by h and h' and by integration by parts, as in the method in the evolution case. This shows that $S(h, h') - S(f, f') \le 0$, which implies $d(h_1, h_0) \le d(f_1, f_0)$.

3. Proof of the results

Proof of Theorem 1.1. We first apply the method to smooth positive solutions. In this case, the heuristic reasoning is rigorous. Then we deduce the result for general initial data, by approximation (see the classical results in [11,12,9]). □

Proof of Theorem 1.2. Given a smooth positive function f, and using standard elliptic theory, it is easy to construct a smooth solution h that satisfies (from the maximum principle): $\min_\Omega f \le h \le \max_\Omega f$. In this framework, the heuristic method is rigorous and gives the result. We then recover the result for general data, by a standard approximation argument (see classical results, for instance, in [1]). □

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