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Completeness on locally convex cones

Intégralité de cônes localement convexes

Mohammad Reza Motallebi

Department of Mathematics, Faculty of Mathematical Sciences, University of Mohaghegh Ardabili, Ardabil, P.O. Box 179, Iran

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ABSTRACT

We investigate complete and compact subsets for the lower, upper and symmetric topologies of a locally convex cone and prove that weakly closed sets will be weakly compact, whenever they are weakly precompact. This leads to the weak* compactness of the polars of neighborhoods and weak compactness of the lower, upper and symmetric neighborhoods.

RÉSUMÉ

Nous étudions des sous-ensembles complets et compacts pour le bas, le haut et les topologies symétriques d'un cône localement convexe, et prouvons que les ensembles faiblement fermés sont faiblement compacts à chaque fois qu'ils sont faiblement pré-compacts. Cela conduit à la faible* compacité des polaires des quartiers et à la faible compacité des quartiers inférieur, supérieur et symétrique.

1. Introduction

The theory of locally convex cones is based on order theoretical notions from which certain topological structures are defined. With these structures, a cone carries three topologies called *lower, upper* and *symmetric* topologies. In particular, a locally convex (ordered) topological vector space is a locally convex cone, and the above three topologies coincide with the given topology. Many basic order and topological properties have been established so far for locally convex cones (see for example [1,3–5]). The aim of this paper is the study of completeness and compactness for the above-mentioned topologies. In Section 2, we study the relations between compact and complete subsets. In Section 3, we discuss the weak completeness and compactness of weakly closed sets and show that the polar of each neighborhood is weak* compact in the upper topology, and for its compactness in the lower (symmetric) topology, the upper (respectively, symmetric) precompactness is necessary. Also, using the strict separation property, one can show that every lower (upper and symmetric) precompact neighborhood is weakly upper (respectively, lower and symmetric) compact.

An *ordered cone* is a set $\mathcal{P}$ endowed with an addition $(a, b) \longmapsto a + b$ and a scalar multiplication $(\alpha, a) \longmapsto \alpha a$ for real numbers $\alpha \geq 0$. The addition is supposed to be associative and commutative, there is a neutral element $0 \in \mathcal{P}$, and for

E-mail address: motallebi@uma.ac.ir.

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the scalar multiplication, the usual associative and distributive properties hold, that is, $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha + \beta)a = \alpha a + \beta a$, $\alpha(a + b) = \alpha a + \alpha b$, $1a = a$, $0a = 0$ for all $a, b \in \mathcal{P}$ and $\alpha, \beta \geq 0$. In addition, the cone $\mathcal{P}$ carries a (partial) order, i.e., a reflexive transitive relation $\leq$ that is compatible with the algebraic operations, that is $a \leq b$ implies $a + c \leq b + c$ and $\alpha a \leq \alpha b$ for all $a, b, c \in \mathcal{P}$ and $\alpha \geq 0$. For example, the extended scalar field $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ of real numbers is a preordered cone. We consider the usual order and algebraic operations in $\overline{\mathbb{R}}$; in particular, $\alpha + \infty = +\infty$ for all $\alpha \in \overline{\mathbb{R}}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0 \cdot (+\infty) = 0$. In any cone $\mathcal{P}$, equality is obviously such an order, hence all results about ordered cones apply to cones without order structures as well.

A *full locally convex cone* $(\mathcal{P}, \mathcal{V})$ is an ordered cone $\mathcal{P}$ that contains an *abstract neighborhood system* $\mathcal{V}$, i.e., a subset of positive elements that is directed downward, closed for addition and multiplication by (strictly) positive scalars. The elements v of $\mathcal{V}$ define *upper* (*lower*) *neighborhoods* for the elements of $\mathcal{P}$ by $v(a) = \{b \in \mathcal{P} : b \leq a + v\}$ (respectively, $(a)v = \{b \in \mathcal{P} : a \leq b + v\}$), creating the *upper*, respectively *lower topologies* on $\mathcal{P}$. Their common refinement is called the *symmetric topology.* We assume all elements of $\mathcal{P}$ to be *bounded below*, i.e., for every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we have $0 \leq a + \rho v$ for some $\rho > 0$. Finally, a *locally convex cone* $(\mathcal{P}, \mathcal{V})$ is a subcone of a full locally convex cone, not necessarily containing the abstract neighborhood system $\mathcal{V}$.

For a locally convex cone $(\mathcal{P}, \mathcal{V})$, the collection of all sets $\widetilde{v} \subseteq \mathcal{P}^2$, where $\widetilde{v} = \{(a, b) : a \leq b + v\}$ for all $v \in \mathcal{V}$, defines a *convex quasi-uniform structure* on $\mathcal{P}$. On the other hand, every convex quasi-uniform structure leads to a full locally convex cone, including $\mathcal{P}$ as a subcone, and induces the same convex quasi-uniform structure. For details, see [1, Ch. I, 5.2].

2. Completeness

Let $(x_\alpha)_{\alpha\in\mathcal{I}}$ be a net in $(\mathcal{P}, \mathcal{V})$ and $x \in \mathcal{P}$. We write $x_\alpha \downarrow x$ ($x_\alpha \uparrow x$) if $(x_\alpha)_{\alpha\in\mathcal{I}}$ converges to x with respect to the lower (respectively, upper) topology. Also $x_\alpha \to x$ means that $x_\alpha \uparrow x$ and $x_\alpha \downarrow x$, i.e., $(x_\alpha)_{\alpha\in\mathcal{I}}$ converges to x with respect to the symmetric topology. We call $(x_\alpha)_{\alpha\in\mathcal{I}}$ in $(\mathcal{P}, \mathcal{V})$ to be *lower* (*upper*) *Cauchy* if, for every $v \in \mathcal{V}$, there is some $\alpha_v \in \mathcal{I}$ such that $x_\beta \leq x_\alpha + v$ (respectively, $x_\alpha \leq x_\beta + v$) for all α, β with $\beta \geq \alpha \geq \alpha_v$. Also $(x_\alpha)_{\alpha\in\mathcal{I}}$ is called *symmetric Cauchy* if it is lower and upper Cauchy, i.e., if, for each $v \in \mathcal{V}$, there is some $\alpha_v \in \mathcal{I}$ such that $x_\beta \leq x_\alpha + v$ and $x_\alpha \leq x_\beta + v$ for all α, β with $\alpha, \beta \geq \alpha_v$. A locally convex cone $(\mathcal{P}, \mathcal{V})$ is called *lower* (*upper* and *symmetric*) complete if every lower (respectively, upper and symmetric) Cauchy net converges in the lower (respectively, upper and symmetric) topology. In general, a set $A \subset \mathcal{P}$ is called lower (upper and symmetric) complete if every lower (respectively, upper and symmetric) Cauchy net is convergent to an element of A in the corresponding topology. Note that if a locally convex cone is symmetric complete, it is not necessary it be both lower and upper complete. For details see [3, Example 2.9].

Recall that a net $(y_\lambda)_{\lambda\in\Lambda}$ is a *subnet* of a net $(x_\alpha)_{\alpha\in\mathcal{I}}$ if there exists a function $\varphi : \Lambda \to \mathcal{I}$ such that $y_\lambda = x_{\varphi_\lambda}$ for each $\lambda \in \Lambda$, where φ_λ stands for $\varphi(\lambda)$; and for each $\alpha_0 \in \mathcal{I}$ there exists some $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies $\varphi_\lambda \geq \alpha_0$. A net $(x_\alpha)_{\alpha\in\mathcal{I}}$ in a set X is said to be an *ultranet* if, for every subset A of X, either $(x_\alpha)_{\alpha\in\mathcal{I}}$ is eventually in A or $(x_\alpha)_{\alpha\in\mathcal{I}}$ is eventually in $X \setminus A$. Every net has a subnet which is an ultranet [6, Ch. 4, 11]. If an ultranet $(x_\alpha)_{\alpha\in\mathcal{I}}$ is eventually in $\bigcup_{i=1}^n A_i$, then it will be eventually in A_i for some i. For, if not, $(x_\alpha)_{\alpha\in\mathcal{I}}$ will be eventually in $X \setminus A_i$ for each i, and so $x_\alpha \in (\bigcup_{i=1}^n A_i) \cup (X \setminus \bigcup_{i=1}^n A_i)$ eventually, which is a contradiction.

We say that a set $A \subset (\mathcal{P}, \mathcal{V})$ is *symmetric precompact* (or *uniformly precompact*), if for every $v \in \mathcal{V}$, there are elements $a_1, \ldots, a_n \in A$ such that $A \subset \bigcup_{i=1}^n v(a_i)v$, i.e., if for each $v \in \mathcal{V}$ there exist subsets $A_1, \ldots, A_n \subset \mathcal{P}$ such that $A \subset \bigcup_{i=1}^n A_i$ and $A_i \times A_i \subset \widetilde{v}$, $i = 1, 2, \ldots, n$. Also, a set $A \subset (\mathcal{P}, \mathcal{V})$ is called *upper* (*lower*) *precompact*, if for every $v \in \mathcal{V}$ and every subset S of A there are elements $s_1, \ldots, s_n \in S$ such that $A \subset \bigcup_{i=1}^n v(s_i)$ (respectively, $A \subset \bigcup_{i=1}^n (s_i)v$). Note that every symmetric precompact set is both lower and upper precompact. But the lower or upper precompact sets need not be the symmetric precompact; for example in $(\overline{\mathbb{R}}, \varepsilon)$, where $\varepsilon = \{\epsilon : \epsilon > 0\}$ the intervals $(-\infty, a]$, $a \in \overline{\mathbb{R}}$ are upper precompact but not lower or symmetric precompact. Obviously, every subset of a symmetric (upper or lower) precompact set is again symmetric (upper or lower) precompact.

Lemma 2.1. *For $A \subseteq \mathcal{P}$, we have*

(*a*) *A is lower precompact if and only if every ultranet in A has an upper Cauchy subnet,*
(*b*) *A is upper precompact if and only if every ultranet in A has a lower Cauchy subnet,*
(*c*) *A is symmetric precompact if and only if every ultranet in A is symmetric Cauchy.*

Proof. (*a*) Let A be lower precompact and $(x_\alpha)_{\alpha\in\mathcal{I}}$ be an ultranet in A. If, for each $\alpha_0 \in \mathcal{I}$, we put $S_{\alpha_0} = \{x_\alpha : \alpha \geq \alpha_0\}$, then S_{α_0} will be lower precompact; so, for each $\alpha_0 \in \mathcal{I}$ and $v \in \mathcal{V}$, there is a finite subset $\Delta_{\alpha_0,v}$ of $\mathcal{I}$ such that $S_{\alpha_0} \subset \bigcup_{\alpha\in\Delta_{\alpha_0,v}} (x_\alpha)v$, which implies that $S_{\alpha_v} \subset (x_{\alpha_v})v$ for some $\alpha_v \in \Delta_{\alpha_0,v}$. We set $\Lambda = \{(\alpha_v, (x_{\alpha_v})v) : \alpha_v \in \Delta_{\alpha_0,v}$ where $\alpha_0 \in \mathcal{I}, v \in \mathcal{V}\}$ and order Λ as follows $(\alpha_v, (x_{\alpha_v})v) \leq (\alpha_u, (x_{\alpha_u})u)$ if and only if $\alpha_v \leq \alpha_u$ and $u \leq v$. This is easily verified to be a direction on Λ and the function $\varphi : \Lambda \to \mathcal{I}$ such that $\varphi((\alpha_v, (x_{\alpha_v})v)) = \alpha_v$ for all $v \in \mathcal{V}$ defines a subnet $(x_{\alpha_v})_{v\in\mathcal{V}}$ of $(x_\alpha)_{\alpha\in\mathcal{I}}$. Suppose $u, v, w \in \mathcal{V}$ such that $\alpha_w \geq \alpha_u \geq \alpha_v$. Then we have $w \leq u \leq v$ and $S_{\alpha_w} \subseteq S_{\alpha_u} \subseteq S_{\alpha_v}$, which implies that $x_{\alpha_u} \leq x_{\alpha_w} + u \leq x_{\alpha_w} + v$, i.e., $(x_{\alpha_v})_{v\in\mathcal{V}}$ is upper Cauchy. Conversely, suppose that every ultranet in A has an upper Cauchy subnet, but *that* A is not lower precompact. There is $v \in \mathcal{V}$ such that A has no finite covering $\bigcup_{i=1}^n (a_i)v$ with $a_i \in A$, $n \in \mathbb{N}$. Then we can find a sequence $(a_n)_{n\in\mathbb{N}}$ in A such that for each $n \in \mathbb{N}$, $a_{n+1} \notin \bigcup_{i=1}^n (a_i)v$. Clearly, $(a_n)_{n\in\mathbb{N}}$ does not have an upper Cauchy subnet, which is a contradiction. In the similar way, we prove (b).

Half of part (c) is similar to (a). Conversely, suppose that A is symmetric precompact and let $(x_\alpha)_{\alpha\in\mathcal{I}}$ be an ultranet in A. For every $v\in\mathcal{V}$, there are $a_1,a_2,\dots,a_n\in A$ such that $A\subseteq\bigcup_{i=1}^n v/2(a_i)v/2$, which yields $x_\alpha\in v/2(a_i)v/2$ eventually for some i. That is, there exists some $\alpha_v\in\mathcal{I}$ such that $x_\alpha\in v(x_\beta)v$ for all $\alpha,\beta\in\mathcal{I}$ satisfying $\alpha,\beta\ge\alpha_v$, i.e., $(x_\alpha)_{\alpha\in\mathcal{I}}$ is symmetric Cauchy. □

Theorem 2.2. *For a topological space X, the following are equivalent;*

(a) *$A\subseteq X$ is compact,*
(b) *every net in A has a subnet which is convergent to an element of A,*
(c) *every ultranet in A is convergent to an element of A.*

Proof. See [6, Ch. 6, Theorem 17.4]. □

Theorem 2.3. *Let $(\mathcal{P},\mathcal{V})$ be a locally convex cone and $A\subseteq\mathcal{P}$. Then*

(a) *if A is lower precompact and upper complete, then it will be upper compact,*
(b) *if A is upper precompact and lower complete, then it will be lower compact,*
(c) *if A is lower (upper) compact, then it will be lower (respectively, upper) complete,*
(d) *A is symmetric compact if and only if it is symmetric precompact and symmetric complete.*

Proof. (a) If A is lower precompact and upper complete then, by Lemma 2.1 and Theorem 2.2, it will be upper compact. Similarly, Parts (b) and (d) follow from Lemma 2.1 and Theorem 2.2. For (c), let A be lower compact, $(x_\alpha)_{\alpha\in\mathcal{I}}$ be a lower Cauchy net in A and $(x_{\alpha_\lambda})_{\lambda\in\Lambda}$ be a subnet of $(x_\alpha)_{\alpha\in\mathcal{I}}$, which is an ultranet. By Lemma 2.1, there is some $x\in A$ such that $x_{\alpha_\lambda}\downarrow x$ then, for every $v\in\mathcal{V}$, there is $\lambda_v\in\Lambda$ such that $x\le x_{\alpha_\lambda}+v/2$ for all $\lambda\ge\lambda_v$. Since $(x_\alpha)_{\alpha\in\mathcal{I}}$ is lower Cauchy, there is $\alpha_0\ge\alpha_{\lambda_v}$ such that $x_\beta\le x_\alpha+v/2$ for all $\beta\ge\alpha\ge\alpha_0$. Let $\beta\ge\alpha_0$ and choose $\lambda\in\Lambda$ such that $\lambda\ge\lambda_v$ and $\alpha_\lambda\ge\beta$. Then $x\le x_{\alpha_\lambda}+v/2\le x_\beta+v/2+v/2$, i.e., $x_\alpha\downarrow x$, that is, A is lower complete. □

3. Weak compactness and neighborhoods

A *dual pair* $(\mathcal{P},\mathcal{Q})$ consists of two cones $\mathcal{P}$ and $\mathcal{Q}$ with a bilinear mapping $(a,x)\longmapsto\langle a,x\rangle:\mathcal{P}\times\mathcal{Q}\longrightarrow\overline{\mathbb{R}}$. Let $(\mathcal{P},\mathcal{Q})$ be a dual pair and X be a collection of subsets of $\mathcal{Q}$ such that:

(p_0) $\inf\{\langle a,x\rangle:x\in A\}>-\infty$ for all $a\in\mathcal{P}$ and $A\in X$.
(p_1) $\lambda A\in X$ for all $A\in X$ and $\lambda>0$.
(p_2) for all $A,B\in X$ there is some $C\in X$ such that $A\cup B\subseteq C$.

For each $A\in X$, we define $\mathrm{U}_A=\{(a,b)\in\mathcal{P}\times\mathcal{P}:\langle a,x\rangle\le\langle b,x\rangle+1$ for all $x\in A\}$. The set of all U_A, $A\in X$ forms a convex quasi-uniform structure with property (U5) in [1, Ch. I, 5.2] and defines a locally convex structure on $\mathcal{P}$. This is called the *X-topology* on $\mathcal{P}$. For each $A\in X$, we denote by v_A the abstract neighborhood induced on $\mathcal{P}$ by U_A. Therefore $(a,b)\in\mathrm{U}_A$ if and only if $a\le b+v_A$. For details, see [1, Ch. II, 3].

For every finite subset $B=\{x_1,x_2,\dots,x_n\}$ of $\mathcal{Q}$, let us denote by $\mathcal{Q}_B$ the subcone of $\mathcal{Q}$ generated by B, that is, $\mathcal{Q}_B=\{\sum_{i=1}^m\alpha_ix_i:x_i\in B,\ \alpha_i\ge0,\ 1\le m\le n\}$. If X_B is the set of all finite subsets of $\mathcal{Q}_B$, the resulting X_B-topology on $\mathcal{P}$ is called the *weak topology* $\sigma(\mathcal{P},\mathcal{Q}_B)$. In particular, the weak topologies $\sigma(\mathcal{P},\mathcal{Q}_x)$ for all $x\in\mathcal{Q}$ are the coarsest ones for this duality. If $X=\bigcup_{x\in X}X_x$, then X consists of the all finite subsets of $\mathcal{Q}$ and the X-topology will be the *weak topology* $\sigma(\mathcal{P},\mathcal{Q})$.

The set $A\subset(\mathcal{P},\mathcal{V})$ is called *lower (upper) bounded*, if for every $v\in\mathcal{V}$, there is some $\lambda>0$ such that $0\le A+\lambda v$ (respectively, $A\le\lambda v$). Also, A is called *bounded*, if it is both lower and upper bounded, that is, there is some $\lambda>0$ such that $A\le\lambda v$ and $0\le A+\lambda v$. It is easy to see that, in a dual pair $(\mathcal{P},\mathcal{Q})$, a set $A\subseteq\mathcal{P}$ is $\sigma(\mathcal{P},\mathcal{Q})$-bounded whenever for every finite subset B of $\mathcal{Q}$, $-\infty<\inf\langle A,B\rangle\le\sup\langle A,B\rangle<+\infty$, where $\langle A,B\rangle=\{\langle a,x\rangle:a\in A,x\in B\}$. Likewise, A is $\sigma(\mathcal{P},\mathcal{Q})$-lower (upper) bounded if, for every finite subset B of $\mathcal{Q}$, $\inf\langle A,B\rangle>-\infty$ (respectively, $\sup\langle A,B\rangle<+\infty$). If $A\subseteq\mathcal{P}$ is $\sigma(\mathcal{P},\mathcal{Q})$-symmetric precompact, then it will be $\sigma(\mathcal{P},\mathcal{Q})$-lower bounded. For, let $x\in\mathcal{Q}$. There are $a_1,\dots,a_n\in A$ such that $A\subseteq\bigcup_{i=1}^n v_x(a_i)v_x$. If we choose $\lambda>0$ such that $0\le a_i+\lambda v_x$, then we imply that $0\le A+\lambda v_x$, i.e., A is $\sigma(\mathcal{P},\mathcal{Q})$-lower bounded.

Proposition 3.1. *If $A\subseteq\mathcal{P}$ is $\sigma(\mathcal{P},\mathcal{Q})$-bounded, then it will be $\sigma(\mathcal{P},\mathcal{Q})$-symmetric precompact.*

Proof. See [2, Proposition 2.21]. □

Proposition 3.2. *If $A\subseteq\mathcal{P}$ is $\sigma(\mathcal{P},\mathcal{Q})$-upper (lower) bounded, then it will be $\sigma(\mathcal{P},\mathcal{Q})$-upper (respectively, lower) precompact.*

Proof. Let A be $\sigma(\mathcal{P},\mathcal{Q})$-upper bonded and B be a finite subset of $\mathcal{Q}$. Fix $a_0\in A$ and put $\lambda=\min\langle B,a_0\rangle$. If we put $A_0=\{a\in A:\langle a,a_0\rangle\le\lambda\}$, then $A_0\subseteq v_B(a_0)$, i.e., A_0 is the $\sigma(\mathcal{P},\mathcal{Q}_B)$-upper precompact. Also, $A\setminus A_0=\{a\in A:\langle a,a_0\rangle>\lambda\}$

is $\sigma(\mathcal{P},\mathcal{Q}_B)$-lower bounded, hence by Proposition 3.1, it will be $\sigma(\mathcal{P},\mathcal{Q}_B)$-upper precompact. Thus, as the union of two $\sigma(\mathcal{P},\mathcal{Q}_B)$-upper precompact sets, A will be $\sigma(\mathcal{P},\mathcal{Q}_B)$-upper precompact for all finite subsets B of $\mathcal{Q}$ and so it is $\sigma(\mathcal{P},\mathcal{Q})$-upper precompact. □

Let $(\mathcal{P},\mathcal{Q})$ be a dual pair and X be a collection of subsets of $\mathcal{Q}$ satisfying (p_0), (p_1) and (p_2). If $(x_\alpha)_{\alpha\in\mathcal{I}}$ is a net in $(\mathcal{P},\mathcal{V}_X)$ and $a\in\mathcal{P}$, then $x_\alpha\uparrow a$ (respectively, $x_\alpha\downarrow a$ and $x_\alpha\to a$) in $(\mathcal{P},\mathcal{V}_X)$ if and only if $\langle x_\alpha,x\rangle\uparrow\langle a,x\rangle$ (respectively, $\langle x_\alpha,x\rangle\downarrow\langle a,x\rangle$ and $\langle x_\alpha,x\rangle\to\langle a,x\rangle$) in $(\overline{\mathbb{R}},\varepsilon)$ for all $x\in A$ and $A\in X$. For, if $x_\alpha\uparrow a$ and $A\in X$, there is some $\alpha_A\in\mathcal{I}$ such that $x_\alpha\le a+v_A$ for all $\alpha\ge\alpha_A$ or $\langle x_\alpha,x\rangle\le\langle a,x\rangle+1$ for all $x\in A$, i.e., $\langle x_\alpha,a\rangle\uparrow\langle a,x\rangle$ in $(\overline{\mathbb{R}},\varepsilon)$ for all $x\in A$. In particular, if $\mathcal{Q}=\bigcup_{A\in X}A$, then, for all $a\in\mathcal{P}$, $x_\alpha\uparrow a$ in X-topology if and only if $\langle x_\alpha,a\rangle\uparrow\langle x,a\rangle$ in $(\overline{\mathbb{R}},\varepsilon)$ for all $x\in\mathcal{P}$.

Proposition 3.3. *If $(\mathcal{P},\mathcal{Q})$ is a dual pair and $A\subseteq\mathcal{P}$, then*

(a) *if A is closed with respect to the $\sigma(\mathcal{P},\mathcal{Q})$-lower (upper) topology, then it will be $\sigma(\mathcal{P},\mathcal{Q})$-upper complete,*
(b) *if A is $\sigma(\mathcal{P},\mathcal{Q})$-lower bounded and closed with respect to the $\sigma(\mathcal{P},\mathcal{Q})$-upper (lower) topology, then it will be $\sigma(\mathcal{P},\mathcal{Q})$-lower complete,*
(c) *if A is closed with respect to the $\sigma(\mathcal{P},\mathcal{Q})$-symmetric topology, then it will be $\sigma(\mathcal{P},\mathcal{Q})$-symmetric complete.*

Proof. According to [2, Proposition 2.10], it is enough to prove the statements of the theorem for $\sigma(\mathcal{P},\mathcal{Q}_x)$-topologies for all $x\in\mathcal{Q}$. Let us denote by $\overline{A}_x$, $\overline{\overline{A}}_x$ and $\overline{A}^s_x$, the closure of A in the lower, upper and symmetric topology of $\sigma(\mathcal{P},\mathcal{Q}_x)$, respectively. For (a), let $(y_\alpha)_{\alpha\in\mathcal{I}}$ be a net in A. If, for each $x\in\mathcal{Q}$, we set $\langle y,x\rangle=\sup_{\alpha\in\mathcal{I}}\langle y_\alpha,x\rangle$ then $y_\alpha\le y+v_x$ for all $\alpha\in\mathcal{I}$, that is, $y_\alpha\uparrow y$ in $\sigma(\mathcal{P},\mathcal{Q}_x)$, hence $y\in\overline{\overline{A}}_x$. Fix $x\in\mathcal{Q}$. If $\sup_{\alpha\in\mathcal{I}}\langle y_\alpha,x\rangle=+\infty$, then $\langle y_\alpha,x\rangle=+\infty$ for some $\alpha\in\mathcal{I}$, which yields $y\le y_\alpha+v_x$, that is, $y\in\overline{A}_x$. Suppose that $\sup_{\alpha\in\mathcal{I}}\langle y_\alpha,x\rangle<+\infty$. There is some $\alpha_x\in\mathcal{I}$ such that $\langle y_{\alpha_x},x\rangle>\langle y,x\rangle-1$, i.e., $y\le y_{\alpha_x}+v_x$, that is, $y\in\overline{A}_x$. Part (b) is proved in the similar way. For (c), let $(y_\alpha)_{\alpha\in\mathcal{I}}$ be a symmetric Cauchy net in A. For every $x\in\mathcal{Q}$, there is some $\alpha_{v_x}\in\mathcal{I}$ such that $y_\alpha\le y_\beta+v_x$ and $y_\beta\le y_\alpha+v_x$ for all $\alpha,\beta\in\mathcal{I}$ with $\alpha,\beta\ge\alpha_{v_x}$. This implies that $-\infty<\inf_{\alpha\in\mathcal{I}}\langle y_\alpha,x\rangle\le\sup_{\alpha\in\mathcal{I}}\langle y_\alpha,x\rangle\le\inf_{\alpha\in\mathcal{I}}\langle y_\alpha,x\rangle<+\infty$. If we set $\langle y,x\rangle=\sup_{\alpha\in\mathcal{I}}\langle y_\alpha,x\rangle=\inf_{\alpha\in\mathcal{I}}\langle y_\alpha,x\rangle$, then $y_\alpha\to y$ in $\sigma(\mathcal{P},\mathcal{Q})$, hence $y\in\overline{A}^s_x$. □

As a consequence of Theorem 2.3 and Proposition 3.3, we have:

Theorem 3.4. *If $(\mathcal{P},\mathcal{Q})$ is a dual pair and $A\subseteq\mathcal{Q}$, then*

(a) *if A is closed in the $\sigma(\mathcal{P},\mathcal{Q})$-lower (upper) topology and is $\sigma(\mathcal{P},\mathcal{Q})$-lower precompact, then it will be $\sigma(\mathcal{P},\mathcal{Q})$-upper compact,*
(b) *if A is lower bounded and is closed in the $\sigma(\mathcal{P},\mathcal{Q})$-upper (lower) topology and is $\sigma(\mathcal{P},\mathcal{Q})$-upper precompact, then it will be $\sigma(\mathcal{P},\mathcal{Q})$-lower compact,*
(c) *if A is closed in the $\sigma(\mathcal{P},\mathcal{Q})$-symmetric topology and is $\sigma(\mathcal{P},\mathcal{Q})$-symmetric precompact, then it will be $\sigma(\mathcal{P},\mathcal{Q})$-symmetric compact.*

In a locally convex cone $(\mathcal{P},\mathcal{V})$, the *polar* v° of $v\in\mathcal{V}$ consists of all linear functionals μ on $\mathcal{P}$ satisfying $\mu(a)\le\mu(b)+1$, whenever $a\le b+v$ for $a,b\in\mathcal{P}$. The union of all polars of neighborhoods forms the *dual cone* $\mathcal{P}^*$ of $\mathcal{P}$. The functionals belonging to $\mathcal{P}^*$ are said to be (*uniformly*) *continuous*. The polar v° of a neighborhood $v\in\mathcal{V}$ is seen to be $w(\mathcal{P}^*,\mathcal{P})$-compact and convex, where $w(\mathcal{P}^*,\mathcal{P})$ denotes the *topology of pointwise convergence* of the elements of $\mathcal{P}$, considered as functions on $\mathcal{P}^*$ with values in $\overline{\mathbb{R}}$ with its usual topology [1, Ch. II, 2.4]. For compactness of v° in the weak topology $\sigma(\mathcal{P}^*,\mathcal{P})$, we have:

Corollary 3.5. *If $(\mathcal{P},\mathcal{V})$ is a locally convex cone and $v\in\mathcal{V}$ then,*

(a) *the polar v° is $\sigma(\mathcal{P}^*,\mathcal{P})$-upper compact,*
(b) *the polar v° will be $\sigma(\mathcal{P}^*,\mathcal{P})$-lower compact, whenever it is $\sigma(\mathcal{P}^*,\mathcal{P})$-upper precompact,*
(c) *the polar v° will be $\sigma(\mathcal{P}^*,\mathcal{P})$-symmetric compact, whenever it is $\sigma(\mathcal{P}^*,\mathcal{P})$-symmetric precompact.*

Proof. It is easy to see that v° is closed in both $\sigma(\mathcal{P}^*,\mathcal{P})$-lower and symmetric topologies. Thus, (b) holds by Theorem 3.4(b). Let B be a finite subset of $\mathcal{P}$ and choose $\lambda>0$ such that $0\le x+\lambda v$ for all $x\in B$. Then $0\le\mu(x)+\lambda$ for all $\mu\in v^\circ$, that is, $0\le v^\circ+\lambda v_B$ or v° is $\sigma(\mathcal{P}^*,\mathcal{P})$-lower bounded, hence it is precompact by Proposition 3.2. Thus (a) and (c) hold by parts (a) and (c) of Theorem 3.4. □

Remark 1. If all of the elements of $(\mathcal{P},\mathcal{V})$ are bounded, then the polar v° of v will be $\sigma(\mathcal{P}^*,\mathcal{P})$-symmetric compact. For, if B is a finite subset of $\mathcal{P}$, then we have $v^\circ\le\lambda v_B$ for some $\lambda>0$, that is, v° is $\sigma(\mathcal{P}^*,\mathcal{P})$-bounded. Hence, by Proposition 3.1 and Corollary 3.5, it will be $\sigma(\mathcal{P}^*,\mathcal{P})$-symmetric compact.

A locally convex cone $(\mathcal{P},\mathcal{V})$ is said to have the *strict separation property*, if for all $a,b\in\mathcal{P}$ and $v\in\mathcal{V}$ with $a\not\le b+\rho v$ for some $\rho>1$, there is a $\mu\in v^\circ$ such that $\mu(a)>\mu(b)+1$. Every locally convex cone $(\mathcal{P},\mathcal{V})$ has the X-topology, where

$X = \{v^\circ : v \in \mathcal{V}\}$ and $(\mathcal{P}, \mathcal{V})$ will be equivalent to $(\mathcal{P}, \mathcal{V}_X)$, whenever it carries the strict separation property; in fact, $\widetilde{v} \subseteq \mathrm{U}_{v^\circ} \subseteq 2\widetilde{v}$ for all $v \in \mathcal{V}$ [1, II, 3.3]. Hence Corollary 3.5(a) yields:

Corollary 3.6. *If $(\mathcal{P}, \mathcal{V})$ has the strict separation property, then it will be equivalent to the X-topology of all $\sigma(\mathcal{P}^*, \mathcal{P})$-upper compact sets v°, $v \in \mathcal{V}$.*

In the condition of the strict separation property, we find a base for each upper, lower and symmetric topology such that the elements of the base for the upper topology are closed in the lower one, and the elements of the base for the lower topology are closed in the upper one, in particular, the elements of the base for the symmetric topology are closed. Indeed, for every $v \in \mathcal{V}$, we have $v(a) \subseteq v_{v^\circ}(a) \subseteq (2v)(a)$ and $v_{v^\circ}(a)$ is closed in the lower topology; for, if $(x_\alpha)_{\alpha \in \mathcal{I}}$ is a net in $v_{v^\circ}(a)$ such that $x_\alpha \downarrow x$ then we have $\mu(x_\alpha) \downarrow \mu(x)$ for all $\mu \in v^\circ$, since μ is continuous in the lower topology. Thus $\mu(x) \le \mu(a) + 1$ for all $\mu \in v^\circ$, that is, $x \le a + v_{v^\circ}$ or $x \in v_{v^\circ}(a)$. In particular, for an X-topology $(\mathcal{P}, \mathcal{V}_X)$, the lower (upper and symmetric) neighborhoods are closed in the upper (respectively, lower and symmetric) topology. Now, by [2, Proposition 2.10], Corollary 3.6 and Theorem 3.4 we have:

Corollary 3.7. *If $(\mathcal{P}, \mathcal{V})$ has the strict separation property, then*

(a) *the lower precompact neighborhoods of the lower topology are $\sigma(\mathcal{P}, \mathcal{P}^*)$-upper compact,*
(b) *the upper precompact neighborhoods of the upper topology are $\sigma(\mathcal{P}, \mathcal{P}^*)$-lower compact,*
(c) *the symmetric precompact neighborhoods of the symmetric topology are $\sigma(\mathcal{P}, \mathcal{P}^*)$-symmetric compact.*

We may also consider the local lower, upper and symmetric neighborhoods of a locally convex cone $\mathcal{P}$ that arise if we endow $\mathcal{P}$ with the neighborhood subsystem $\mathcal{V}_v = \{\alpha v : \alpha \ge 0\}$ consisting of the multiples of a single neighborhood $v \in \mathcal{V}$. Obviously, the dual cone $\mathcal{P}_v^*$ of $(\mathcal{P}, \mathcal{V}_v)$ consists only of the multiples of the functionals in v° [5, Ch. I, 4]. If $b \in \mathcal{P}$ and $\mu \in \mathcal{P}_v^*$, then $0 \le b + \rho v$ and $\mu \in \rho v^\circ$ for some $\rho > 0$, which yields $0 \le (b)v + \rho^2 v_\mu$ hence, by Proposition 3.2, the lower neighborhood $(b)v$ is $\sigma(\mathcal{P}, \mathcal{P}^*)$-lower precompact. Similarly, if an element b of $\mathcal{P}$ is v-bounded, the upper (symmetric) neighborhoods of b will be $\sigma(\mathcal{P}, \mathcal{P}^*)$-upper (respectively, symmetric) precompact. Hence we have:

Corollary 3.8. *For $v \in \mathcal{V}$, if $(\mathcal{P}, \mathcal{V}_v)$ has the strict separation property, then*

(a) *the lower neighborhoods of $\mathcal{P}$ are $\sigma(\mathcal{P}, \mathcal{P}_v^*)$-upper compact,*
(b) *the upper neighborhoods of the bounded elements of $\mathcal{P}$ are $\sigma(\mathcal{P}, \mathcal{P}_v^*)$-lower compact,*
(c) *the symmetric neighborhoods of the bounded elements of $\mathcal{P}$ are $\sigma(\mathcal{P}, \mathcal{P}_v^*)$-symmetric compact.*

Remark 2. (i) If $(\mathcal{P}, \mathcal{V})$ has the strict separation property and $A \subset \mathcal{P}$ is upper (lower and symmetric) precompact, then the closure of A with respect to the lower (respectively, upper and symmetric) topology will be upper (respectively, lower and symmetric) precompact. For, if A is upper precompact, then, for each $v \in \mathcal{V}$ and each subset S of A, there are $s_1, \ldots, s_n \in S$ such that $S \subset \bigcup_{i=1}^n 1/2v(s_i)$, which implies that $\overline{S} \subseteq \bigcup_{i=1}^n v(s_i)$, since $\overline{1/2v(s_i)} \subseteq \overline{v_{(1/2v)^\circ}(s_i)} \subseteq v(s_i)$, that is, $\overline{A}$ is upper precompact. Therefore, according to [2, Proposition 2.10] and Theorem 3.4, the closure $\overline{\overline{A}}$ will be $\sigma(\mathcal{P}, \mathcal{P}^*)$-upper compact, whenever A is lower bounded, and the closure $\overline{A}$ will be $\sigma(\mathcal{P}, \mathcal{P}^*)$-lower compact, whenever A is both lower bounded and upper precompact; also, if A is bounded or symmetric precompact, then $\overline{A}^s$ will be symmetric compact. In particular, for every $a \in \mathcal{P}$, the closure $\bar{a}$ is upper compact and $\bar{a}^s$ is symmetric compact.

(ii) If $A \subseteq \mathcal{P}^*$ is equicontinuous, the closure of A in the upper topology will be $\sigma(\mathcal{P}^*, \mathcal{P})$-upper compact; for, $A \subseteq v^\circ$ for some $v \in \mathcal{V}$ [2, 2.5], hence by (i) and Proposition 3.1, $\overline{\overline{A}}$ is $\sigma(\mathcal{P}^*, \mathcal{P})$-lower precompact, since v° is $\sigma(\mathcal{P}^*, \mathcal{P})$-lower bounded. Thus, by Theorem 3.4, $\overline{\overline{A}}$ is $\sigma(\mathcal{P}^*, \mathcal{P})$-upper compact.

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