



ELSEVIER

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com



Combinatorics/Algebra

## The algebra of Hurwitz multizeta functions



## Sur l'algèbre des multizêtas de Hurwitz

Olivier Bouillot

Laboratoire d'informatique Gaspard-Monge (LIGM), Université Paris-Est Marne-la-Vallée, Cité Descartes, 5, bd Descartes,  
77454 Marne-la-Vallée cedex 2, France

## ARTICLE INFO

## Article history:

Received 27 December 2013

Accepted after revision 11 September 2014

Available online 26 September 2014

Presented by the Editorial Board

## ABSTRACT

Multizeta values are real numbers that span a complicated algebra: there exist two different interacting products. A functional analog of these numbers is defined so as to obtain a better understanding of them, the Hurwitz multizeta functions, which span an algebra for which a precise description is wanted. In this note, we prove that the algebra of Hurwitz multizeta functions is a polynomial algebra.

© 2014 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

## R É S U M É

Les multizêtas sont des nombres réels possédant une structure d'algèbre complexe : il existe deux produits interagissant. Il est naturel de définir un analogue fonctionnel de ces nombres pour en avoir une meilleure compréhension, ce qui conduit aux multizêtas de Hurwitz, dont on souhaiterait connaître précisément la structure d'algèbre. Dans cette note, nous montrons que l'algèbre des multizêtas de Hurwitz est une algèbre de polynômes.

© 2014 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

## Version française abrégée

Pour toute séquence  $(s_1, \dots, s_r) \in (\mathbb{N}^*)^r$ , de longueur  $r \in \mathbb{N}^*$  telle que  $s_1 \geq 2$ , on appelle multizêta de Hurwitz (ou encore polyzêta monocentré de Hurwitz) la fonction définie sur  $\mathbb{C} - (-\mathbb{N})$ , par :

$$\mathcal{H}e^{s_1, \dots, s_r} : z \mapsto \sum_{0 < n_1 < \dots < n_r} \frac{1}{(n_1 + z)^{s_1} \dots (n_r + z)^{s_r}} \quad (1)$$

La condition  $s_1 \geq 2$  assure la convergence de la série ainsi que le caractère holomorphe de  $\mathcal{H}e^{s_1, \dots, s_r}$  sur  $\mathbb{C} - (-\mathbb{N})$ . On définit aussi  $\mathcal{H}e^\emptyset : z \mapsto 1$  (cf. [2] et [3] pour un contexte où ces fonctions apparaissent naturellement en dynamique holomorphe, en lien avec les multitangentes; cf. [8,12] et [13] pour d'autres travaux se rapportant à ces fonctions).

E-mail address: [olivier.bouillot@univ-mlv.fr](mailto:olivier.bouillot@univ-mlv.fr).URL: <http://www-igm.univ-mlv.fr/~bouillot/>.<http://dx.doi.org/10.1016/j.crma.2014.09.006>

1631-073X/© 2014 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Une telle fonction est une évaluation d'une fonction quasi symétrique monomiale  $M_{s_1, \dots, s_r}$  (cf. [9] pour la première présentation de ces fonctions et [1] pour une présentation plus récente). Ainsi, cette famille de fonctions engendre une algèbre notée  $\mathcal{H}mzf_{cv}$ , son produit étant le produit de  $QSym$ , appelé aussi bien *stuffle*, *shuffle augmenté*, produit de battage contractant, etc., suivant les auteurs (cf. [10,7,3], ...). L'objectif de cette note est de décrire la structure algébrique de  $\mathcal{H}mzf_{cv}$ .

On peut remarquer que la famille des multizêtas de Hurwitz est un analogue fonctionnel de celle des multizêtas, ces derniers étant définis par  $\mathcal{Z}e^{s_1, \dots, s_r} = \mathcal{H}e^{s_1, \dots, s_r}(0)$  (cf. [6,15] et [16] pour une introduction aux multizêtas). On sait peu de choses sur la nature arithmétique des multizêtas, malgré quelques avancées récentes (cf. [4,5]). Ces nombres engendrent aussi une algèbre notée  $\mathcal{M}zv_{cv}$ , et l'on peut demander une description algébrique de celle-ci. Par exemple,  $\mathcal{M}zv_{cv}$  est-elle une algèbre graduée, et si oui, quelle est la suite des dimensions de ses composantes homogènes ?

Déterminer la structure algébrique d'une algèbre non formelle, comme  $\mathcal{M}zv_{cv}$  ou  $\mathcal{H}mzf_{cv}$ , est en général une question difficile car, a priori, d'autres produits peuvent entrer en jeu et complexifier la structure de l'algèbre étudiée. C'est en particulier le cas de  $\mathcal{M}zv_{cv}$ . Dans le cas des multizêtas de Hurwitz, la situation est en fait incomparablement plus simple, car le *stuffle*, provenant de l'identification des multizêtas de Hurwitz avec une spécialisation des fonctions quasi symétriques monomiales, est le seul produit possible.

Dans cette note, nous montrons que  $\mathcal{H}mzf_{cv}$  est en fait une algèbre polynomiale. Le point clé est le [Théorème 0.1](#).

**Théorème 0.1.** La famille  $(\mathcal{H}e^{s_1, \dots, s_r})_{\substack{r \in \mathbb{N} \\ s_1, \dots, s_r \in \mathbb{N}^* \\ s_1 \geq 2}}$  est  $\mathbb{C}(z)$ -linéairement indépendante.

Notons que la référence [11] propose une preuve d'une propriété plus faible, l'indépendance linéaire sur  $\mathbb{C}$  des multizêtas de Hurwitz.

Les corollaires suivants de ce théorème permettent de répondre à toutes les questions que l'on peut poser concernant les multizêtas d'Hurwitz. Par exemple :

**Corollaire 0.2.** Notons  $QSym_{cv}$  la sous-algèbre de  $QSym$  engendrée par les monômes  $M_I$  où  $I = (i_1, \dots, i_r)$  est une composition telle que  $i_1 \geq 2$ . Alors :

$$\mathcal{H}mzf_{cv} \simeq QSym_{cv} \simeq \mathbb{Q}\langle \mathcal{L}yn(y_1; y_2; \dots) - \{y_1\} \rangle, \tag{2}$$

où  $\mathcal{L}yn(y_1; y_2; \dots)$  représente l'ensemble des mots de Lyndon sur l'alphabet  $Y = \{y_1; y_2; \dots\}$ .

Pour plus d'informations sur les mots de Lyndon, on renvoie le lecteur à [14].

**Corollaire 0.3.** Toute relation algébrique de  $\mathcal{H}mzv_{cv}$  est conséquence de l'évaluation du produit de *stuffle*.

**Corollaire 0.4.** Notons  $\mathcal{H}mzv_{cv,n}$  l'algèbre des multizêtas de Hurwitz de poids  $n$ , c'est-à-dire la sous-algèbre de  $\mathcal{H}mzv_{cv}$  engendrée par les multizêtas de Hurwitz  $\mathcal{H}e^{s_1, \dots, s_r}$  vérifiant  $s_1 + \dots + s_r = n$ .

Alors :

(i)  $\mathcal{H}mzv_{cv}$  est graduée par le poids :

$$\begin{cases} \mathcal{H}mzv_{cv} = \bigoplus_{n \in \mathbb{N}} \mathcal{H}mzv_{cv,n}. \\ \forall (p, q) \in \mathbb{N}^2, \quad \mathcal{H}mzv_{cv,p} \cdot \mathcal{H}mzv_{cv,q} \subset \mathcal{H}mzv_{cv,p+q}. \end{cases} \tag{3}$$

(ii)

$$\begin{cases} \dim \mathcal{H}mzf_{cv,0} = 1, & \dim \mathcal{H}mzf_{cv,0} = 0. \\ \dim \mathcal{H}mzf_{cv,n} = 2^{n-1} & \text{for all } n \geq 2. \end{cases}$$

### 1. Introduction

For any sequence  $(s_1, \dots, s_r) \in (\mathbb{N}^*)^r$ , of length  $r \in \mathbb{N}^*$  such that  $s_1 \geq 2$ , we define the Hurwitz multizeta function (or mono-center Hurwitz polyzeta), over  $\mathbb{C} - (-\mathbb{N})$ , by:

$$\mathcal{H}e^{s_1, \dots, s_r} : z \mapsto \sum_{0 < n_r < \dots < n_1} \frac{1}{(n_1 + z)^{s_1} \dots (n_r + z)^{s_r}}. \tag{4}$$

The condition  $s_1 \geq 2$  ensures the convergence and, consequently,  $\mathcal{H}e^{s_1, \dots, s_r}$  is a holomorphic function over  $\mathbb{C} - (-\mathbb{N})$ . We also define  $\mathcal{H}e^\emptyset : z \mapsto 1$ . (See [2] and [3], where these functions appear naturally in connection to holomorphic dynamics and multitangent functions; see also [8,12] and [13] for other articles dealing with these functions.)

Such a function is an evaluation of a monomial quasi-symmetric function  $M_{s_1, \dots, s_r}$  (see [9], or [1] for a more recent presentation). Therefore, this family spans an algebra  $\mathcal{H}mzf_{cv}$ , where the product is the product of  $QSym$ , known as the stuffle product (also called augmented shuffle, contracting shuffle, quasi-shuffle... (see [10,7,3], etc.). The aim of this note is to describe the algebraic structure of  $\mathcal{H}mzf_{cv}$ .

The family of Hurwitz multizeta functions is a functional analogue of multizeta values, which are defined by  $\mathcal{Z}e^{s_1, \dots, s_r} = \mathcal{H}e^{s_1, \dots, s_r}(0)$  (see [6,15] and [16]). Little is known on the arithmetical properties of multizeta values, despite the recent works [4,5]. These numbers span an algebra denoted by  $\mathcal{M}zv_{cv}$ . An algebraic description of it may be asked (for instance, if it is a graded algebra, and then a formula for the dimensions of homogeneous components, etc.).

Elucidating the algebraic structure of an algebra, like  $\mathcal{M}zv_{cv}$  or  $\mathcal{H}mzf_{cv}$ , is in general a difficult question, since no one can be sure that there does not exist a second natural product hidden somewhere inside the algebraic structure itself. This is for instance the case in  $\mathcal{M}zv_{cv}$ . In the case of Hurwitz multizeta functions, the situation is considerably simpler, because the stuffle product, coming from the identification of the Hurwitz multizeta functions with a specialization of monomial quasi-symmetric functions, is actually the only product to be taken in account.

In this note, we will explain the reasons that elucidate completely the case of  $\mathcal{H}mzf_{cv}$ . The key point of the proof, which will be the heart of this note, is the following.

**Theorem 1.1.** *The family  $(\mathcal{H}e^{s_1, \dots, s_r})_{\substack{r \in \mathbb{N} \\ s_1, \dots, s_r \in \mathbb{N}^* \\ s_1 \geq 2}}$  is  $\mathbb{C}(z)$ -linearly independent.*

Let us notice that [11] proposes a proof of a weaker property, the linear independence over  $\mathbb{C}$  of the Hurwitz multizeta functions.

The following corollaries answer all the questions that one would ask for the multizeta values.

**Corollary 1.2.** *Let us denote by  $QSym_{cv}$  the subalgebra of  $QSym$  spanned by the monomials  $M_I$  with a composition  $I = (i_1, \dots, i_r)$  such that  $i_1 \geq 2$ . Then:*

$$\mathcal{H}mzf_{cv} \simeq QSym_{cv} \simeq \mathbb{Q}(\mathcal{L}yn(y_1; y_2; \dots) - \{y_1\}), \tag{5}$$

where  $\mathcal{L}yn(y_1; y_2; \dots)$  denotes the set of Lyndon words over the alphabet  $Y = \{y_1; y_2; \dots\}$ .

For more information about Lyndon words, we refer the reader to [14].

**Corollary 1.3.** *Each algebraic relation in  $\mathcal{H}mzv_{cv}$  comes from the expansion of stuffle products.*

**Corollary 1.4.** *Let us denote the algebra of Hurwitz multizeta functions of weight  $n$  by  $\mathcal{H}mzv_{cv,n}$ , that is, the subalgebra of  $\mathcal{H}mzv_{cv}$  spanned by the Hurwitz multizetas  $\mathcal{H}e^{s_1, \dots, s_r}$  such that  $s_1 + \dots + s_r = n$ .*

Then:

(i)  $\mathcal{H}mzv_{cv}$  is graded by the weight:

$$\begin{cases} \mathcal{H}mzv_{cv} = \bigoplus_{n \in \mathbb{N}} \mathcal{H}mzv_{cv,n}. \\ \forall (p, q) \in \mathbb{N}^2, \quad \mathcal{H}mzv_{cv,p} \cdot \mathcal{H}mzv_{cv,q} \subset \mathcal{H}mzv_{cv,p+q}. \end{cases} \tag{6}$$

(ii)  $\begin{cases} \dim \mathcal{H}mzf_{cv,0} = 1, & \dim \mathcal{H}mzf_{cv,0} = 0. \\ \dim \mathcal{H}mzf_{cv,n} = 2^{n-1} & \text{for all } n \geq 2. \end{cases}$

## 2. A fundamental lemma

Hurwitz multizeta functions are “translations” of multizeta values. Therefore, it is natural to examine how the shift operator acts on such functions. It turns out that the Hurwitz multizeta functions satisfy a fundamental difference equation.

**Lemma 2.1.** *For all  $r \in \mathbb{N}$ ,  $(s_1, \dots, s_r) \in (\mathbb{N}^*)^r$  such that  $s_1 \geq 2$ , we have:*

$$\Delta_-(\mathcal{H}e^{s_1, \dots, s_r})(z) = \frac{1}{z^{s_r}} \mathcal{H}e^{s_1, \dots, s_{r-1}}(z), \quad \text{where } \Delta_-(f)(z) = f(z-1) - f(z). \tag{7}$$

### 3. An example of linear independence of Hurwitz multizeta functions

As an example, let us consider three rational functions  $F_2, F_{2,1}$  and  $G$  valued in  $\mathbb{C}$  such that

$$F_2 \mathcal{H}e^2 + F_{2,1} \mathcal{H}e^{2,1} = G. \tag{8}$$

If  $F_{2,1} \neq 0$ , we can assume that  $F_{2,1} = 1$ . Then, applying the operator  $\Delta_-$  to this relation, we obtain:

$$\tilde{F}_2 \mathcal{H}e^2 = \tilde{G}, \tag{9}$$

where

$$\begin{cases} \tilde{F}_2(z) = \Delta_-(F_2)(z) + \frac{1}{z} = \Delta_-(F_2 + \mathcal{H}e^1)(z) \\ \tilde{G}(z) = \Delta_-(G)(z) - \frac{1}{z^2} \cdot F_2(z-1) \end{cases} \quad \text{and} \quad \mathcal{H}e^1(z) = \sum_{n>0} \left( \frac{1}{n+z} - \frac{1}{n} \right). \tag{10}$$

From (9), we can easily deduce that  $\tilde{F}_2 = \tilde{G} = 0$ . Consequently,  $F_2 + \mathcal{H}e^1$  is a 1-periodic function. But, looking at the possible poles, we can conclude that this is not possible. Therefore,  $F_{2,1} = 0$ , which also implies  $F_2 = G = 0$ .

Thus, the function  $1, \mathcal{H}e^2$  and  $\mathcal{H}e^{2,1}$  are  $\mathbb{C}(z)$ -linearly independent.

This argument will be generalized in the proof of Theorem 1.1 and can be enlightened by the following lemma.

**Lemma 3.1.** *Let  $F$  be a rational function and  $f$  a 1-periodic function.*

*If, for a  $n$ -tuple  $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ ,  $F + \sum_{i=1}^n \lambda_i \mathcal{H}e^i = f$  holds, then we necessarily have:*

$$\begin{cases} \lambda_1 = \dots = \lambda_n = 0, \\ F \text{ and } f \text{ are constant functions.} \end{cases}$$

### 4. Linear independence of Hurwitz multizeta functions over the rational function field

We will give a sketch of proof of Theorem 1.1 based on an induction process. Let us introduce some notations, an order and properties.

- a. For all  $d \in \mathbb{N}$ , let  $\mathcal{S}_{\leq d}^*$  and  $\mathcal{S}_d^*$  be the sets defined by  $\mathcal{S}_{\leq d}^* = \{\underline{s} \in \mathcal{S}^*; d^\circ \underline{s} \leq d\}$  and  $\mathcal{S}_d^* = \{\underline{s} \in \mathcal{S}^*; d^\circ \underline{s} = d\}$ , where  $\mathcal{S}^* = \{(s_1, \dots, s_r) \in (\mathbb{N}^*)^r; s_1 \geq 2\}$  and  $d^\circ \underline{s} = d^\circ(s_1, \dots, s_r) = s_1 + \dots + s_r - r$ .
- b. We can order the sequences of  $\mathcal{S}_{d+1}^*$  by numbering first the sequences of length 1, then those of length 2, etc. For the following proof, let us notice that it will not be necessary to clarify the numbering within a length. So, we will consider the sets  $\mathcal{S}_{d+1}^* = \{\underline{s}^n; n \in \mathbb{N}^*\}$  and, for  $n \in \mathbb{N}$ ,  $S_n = \{\underline{s}^i; 1 \leq i \leq n\}$ .
- c. We will finally consider the following properties:

$$\begin{aligned} \mathcal{D}(d) : & \quad \text{“the family } (\mathcal{H}e^{\underline{s}})_{\underline{s} \in \mathcal{S}_{\leq d}^*} \text{ is } \mathbb{C}(z)\text{-linearly independent.”} \\ \mathcal{P}(d, n) : & \quad \text{“the family } (\mathcal{H}e^{\underline{s}})_{\underline{s} \in \mathcal{S}_{\leq d}^*} \cup (\mathcal{H}e^{\underline{s}})_{\underline{s} \in S_n} \text{ is } \mathbb{C}(z)\text{-linearly independent.”} \end{aligned}$$

In order to prove the theorem, we only need to give details on the heredity of the property  $\mathcal{D}(n)$ . This will also be done by an induction process and the proof boils down to the following implication:

$$\forall (d, n) \in \mathbb{N}^2, \quad \mathcal{P}(d, n) \implies \mathcal{P}(d, n+1).$$

In order to do this, let us assume that Property  $\mathcal{P}(d, n)$  holds for a pair  $(d, n) \in \mathbb{N}^2$  and consider the relation:

$$\sum_{\substack{\underline{s} \in \mathcal{S}_{\leq d}^* \cup S_{n+1} \\ \underline{s} \neq \emptyset}} F_{\underline{s}} \mathcal{H}e^{\underline{s}} = F, \tag{11}$$

where  $F$  and  $F_{\underline{s}}$ ,  $\underline{s} \in \mathcal{S}_{\leq d}^* \cup S_{n+1}$ , are rational functions. Thanks to  $\mathcal{P}(d, n)$ , it is sufficient to prove that  $F_{\underline{s}^{n+1}} = 0$ . Using a proof by contradiction, we can then assume that  $F_{\underline{s}^{n+1}} = 1$ .

*Step 1 : Application of  $\Delta_-$  to relation (11).*

Applying  $\Delta_-$  to (11) and using the induction hypothesis  $\mathcal{P}(d, n)$ , we obtain:

$$\begin{cases} \forall \underline{s} \in (\mathcal{S}_{\leq d}^* \cup S_n) - \{\emptyset\}, \quad \Delta_-(F_{\underline{s}})(z) + \sum_{\substack{k \in \mathbb{N}^* \\ \underline{s} \cdot k \in \mathcal{S}_{\leq d}^* \cup S_n}} F_{\underline{s},k}(z-1) \cdot J^k(z) = 0. \\ \Delta_-(F)(z) = \sum_{k=2}^{d+1} F_k(z-1) J^k(z) + (1 - \delta_{n,0}) F_{d+2}(z-1) J^{d+2}(z). \end{cases} \tag{12}$$

Step 2: A lemma which gives some partial solutions to the system (12).

Denoting by  $\underline{s}^{n+1} = \underline{u} \cdot p$  with  $p \geq 1$  and  $\underline{u} \in \mathcal{S}_{\leq d}^* \cup S_n$ , we can prove that:

**Lemma 4.1.** *Let  $r$  be a positive integer and  $p \geq 2$ .*

*Let us also consider two  $r$ -tuples,  $(n_1; \dots; n_r) \in \mathbb{N}^r$  and  $(k_1; \dots; k_r) \in (\mathbb{N}^*)^r$  such that  $\sum_{i=1}^r (k_i - 1) \leq p - 2$ . Then,  $F_{\underline{u}, k_1, 1^{[n_1]} \dots, k_r, 1^{[n_r]}}$  is null if  $n_r > 0$ , and is constant if  $n_r = 0$ .*

Step 3: Revealing the contradiction.

The system (12), applied to  $\underline{u}$ , gives us:

$$\Delta_- \left( F_{\underline{u}} + \sum_{k=1}^{p-1} F_{\underline{u}, k} \mathcal{H}e^k + \mathcal{H}e^p \right) = 0, \tag{13}$$

where  $F_{\underline{u}, 1}, \dots, F_{\underline{u}, p-1}$  are rational fractions which turn out to be constant according to the Lemma 4.1 and now denoted by  $f_{\underline{u}, 1}, \dots, f_{\underline{u}, p-1}$ . Thus,  $F_{\underline{u}} + \sum_{k=1}^{p-1} f_{\underline{u}, k} \mathcal{H}e^k + \mathcal{H}e^p$  defines a 1-periodic function. The coefficients being not all zero in this relation, this contradicts Lemma 3.1 and concludes the proof.

### References

- [1] F. Bergeron, Algebraic Combinatorics and Coinvariant Spaces, CMS Treatise in Mathematics, CRC Press, Boca Raton, FL, USA, 2009, 221 p.
- [2] O. Bouillot, Invariants analytiques des difféomorphismes et multizêtas, PhD thesis, 2011, Orsay, France, 291 p.
- [3] O. Bouillot, The algebra of multitangent functions, J. Algebra 410 (2014) 148–238.
- [4] F. Brown, Mixed Tate motives over  $\mathbb{Z}$ , Ann. of Math. (2) 175 (2) (2012) 949–976.
- [5] F. Brown, On the decomposition of motivic multiple zeta values, in: H. Nakamura, et al. (Eds.), Galois–Teichmüller Theory and Arithmetic Geometry, in: Adv. Stud. Pure Math., vol. 68, 2012, pp. 31–58.
- [6] P. Cartier, Fonctions polylogarithmes, nombres polyzeta et groupes pro-unipotents, in: Séminaire N. Bourbaki, World Scientific, Singapore, 2000–2001, pp. 137–173, exp. 885.
- [7] K. Ebrahimi-Fard, L. Guo, Quasi-shuffles, mixable shuffles and Hopf algebras, J. Algebr. Comb. 24 (2006) 83–101.
- [8] J.-Y. Enjalbert, H.N. Minh, Combinatorial study of colored Hurwitz polyzêtas, Discrete Math. 312 (24) (2013) 3489–3497.
- [9] I. Gessel, Multipartite P-partitions and inner products of skew Schur functions, in: Combinatorics and Algebra, Boulder, CO, USA, 1983, pp. 289–317; Contemp. Math., 34 (1984) 289–301.
- [10] M. Hoffman, Quasi-shuffle products, J. Algebr. Comb. 11 (2000) 49–68.
- [11] S. Joyner, An algebraic version of the Knizhnik–Zamolodchikov equation, Ramanujan J. 28 (3) (2012) 361–384.
- [12] H.N. Minh, G. Jacob, N.E. Oussous, M. Petitot, De l’algèbre des  $\zeta$  de Riemann multivariées à l’algèbre des  $\zeta$  de Hurwitz multivariées, Sémin. Lothar. Comb. 44 (2000), Art. B44i, 21 p. (electronic).
- [13] M.R. Murty, K. Sinha, Multiple Hurwitz zeta function, in: Multiple Dirichlet Series, Automorphic Forms and Analytic Number Theory, in: Proc. Symp. Pure Math., vol. 75, 2006, pp. 135–156.
- [14] C. Reutenauer, Free Lie Algebras, Lond. Math. Soc. Monogr. New Ser., vol. 7, Oxford Sciences Publications, Oxford, UK, 1993.
- [15] M. Waldschmidt, Valeurs zêta multiples. Une introduction, J. Théor. Nr. Bordx. 12 (2000) 581–592.
- [16] V.V. Zudilin, Algebraic relations for multiple zeta values, Russ. Math. Surv. 58 (2003) 1–29.